
QUADRATIC POLYNOMIALS REPRESENTED BY NORM FORMS

by

T.D. Browning & D.R. Heath-Brown

Abstract. — Let $P(t) \in \mathbb{Q}[t]$ be an irreducible quadratic polynomial and suppose that K is a quartic extension of \mathbb{Q} containing the roots of $P(t)$. Let $\mathbf{N}_{K/\mathbb{Q}}(\mathbf{x})$ be a full norm form for the extension K/\mathbb{Q} . We show that the variety

$$P(t) = \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x}) \neq 0$$

satisfies the Hasse principle and weak approximation. The proof uses analytic methods.

Contents

1. Introduction	1
2. Deduction of Theorem 1	7
3. Preliminaries for Theorem 2	10
4. A general approximation principle	15
5. The functions $\hat{\alpha}(x)$ and $\alpha_0(x)$	22
6. A large sieve bound for $\alpha_0(x)$	28
7. Bilinear forms in dimension 2	32
8. Estimation of the main term	38
9. The singular series and integral	46
References	54

1. Introduction

Let k be a number field with set of valuations Ω_k . Given an algebraic variety X defined over k we have the obvious inclusions

$$X(k) \xrightarrow{\Delta} X(\mathbb{A}_k) \subseteq \prod_{\nu \in \Omega_k} X(k_\nu),$$

where Δ is the diagonal embedding of the set $X(k)$ of k -rational points into the set $X(\mathbb{A}_k)$ of adèles of X . Moreover the set $X(\mathbb{A}_k)$ is empty if and only if $\prod_{\nu \in \Omega_k} X(k_\nu)$ is empty and clearly provides a local obstruction to the existence of k -rational points on X . Recall that a class \mathcal{X} of algebraic varieties X defined over k is said to satisfy the Hasse principle if $X(k) \neq \emptyset$

whenever $X(\mathbb{A}_k) \neq \emptyset$. Likewise \mathcal{X} is said to satisfy weak approximation if whenever it is non-empty the image of $X(k)$ under Δ is dense in $X(\mathbb{A}_k)$ in the product of ν -adic topologies.

This paper is concerned with the Hasse principle and weak approximation for the class of varieties satisfying the Diophantine equation

$$P(t) = \mathbf{N}_{K/k}(x_1, \dots, x_n) \neq 0, \quad (1.1)$$

where $\mathbf{N}_{K/k}$ is a full norm form for an extension K/k of number fields, and $P(t)$ is a polynomial over k . Thus if $[K : k] = n$ and we fix a basis $\{\omega_1, \dots, \omega_n\}$ for K as a vector space over k , then

$$\mathbf{N}_{K/k}(x_1, \dots, x_n) := N_{K/k}(x_1\omega_1 + \dots + x_n\omega_n).$$

Throughout this paper we will use $\mathbf{N}_{K/k}$ to denote a norm form, and $N_{K/k}$ to denote the corresponding field norm.

Progress on this problem has been limited, and we begin by discussing what is known in the simplest cases. A crude measure of difficulty is given by the number of distinct roots of $P(t)$ over an algebraic closure \bar{k} . When $P(t)$ is a non-zero constant polynomial the Hasse principle for (1.1) is known as the ‘‘Hasse norm principle’’. The validity of the Hasse norm principle for cyclic extensions K/k was established by Hasse himself, but for non-cyclic extensions there can be counterexamples. There is an extensive literature on the subject and it is known, for example, that the Hasse norm principle holds if the field K has prime degree over k (Bartels [1], for example); or the Galois group of N/k is dihedral, where N is the normal closure of K over k (Bartels [2]); or the extension K/k is Galois and every Sylow subgroup of the Galois group is cyclic (Gurak [16, Corollary 3.2]);.

Following the work of Colliot-Thélène and Sansuc [6, Proposition 9.1], we also have simple sufficient conditions to ensure that ‘‘weak approximation for norms’’ holds, by which we mean that weak approximation holds for (1.1) when $P(t)$ is a non-zero constant polynomial. Let N be the normal closure of K over k . Then weak approximation holds if either the degree $[K : k]$ is prime, or if the Galois group of N/k has cyclic Sylow subgroups. In particular the latter result implies that it suffices for K/k to be cyclic.

The next case to consider is that in which $P(t) = ct^d$ for some $c \in k^\times$ and some positive degree d . In this situation the Hasse principle and weak approximation may fail. However (1.1) is a principal homogeneous space under an algebraic k -torus, and the work of Sansuc [23] and Voskresenskiĭ [24] shows that the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on any smooth projective model of this variety.

When $[K : k] = 2$ and $P(t)$ has degree 3 or 4 then (1.1) defines a Châtelet surface. The arithmetic of such surfaces is well-understood. The Hasse principle and weak approximation may fail, but it has been shown by Colliot-Thélène, Sansuc and Swinnerton-Dyer [9] that all such failures are explained by the Brauer–Manin obstruction. The same conclusion is available when $[K : k] = 3$ and $\deg P(t) \leq 3$, by work of Colliot-Thélène and Salberger [4].

There have also been investigations into (1.1) when $P(t)$ factors completely over k , with at most two roots. In this case one may write

$$P(t) = c(t - a)^u(t - b)^v,$$

with $a, b, c \in k$ and $u, v \in \mathbb{N}$. It is known that one has the Hasse principle and weak approximation whenever the Brauer–Manin obstruction is empty, providing that we work over the ground field $k = \mathbb{Q}$. This was first proved under the assumption that $\gcd(u, v, n) = 1$, by Heath-Brown and Skorobogatov [18], a condition that was subsequently removed by Colliot-Thélène, Harari and Skorobogatov [8]. While all the previous work described had been purely

algebraic, the approach used by Heath-Brown and Skorobogatov combined analytic machinery, in the form of the Hardy–Littlewood circle method, with the previous descent approach to the Brauer–Manin obstruction. The circle method can be adapted, with some effort, to apply to ground fields other than $k = \mathbb{Q}$. However, for simplicity, this possibility was not explored in [18].

Very little is known about other polynomials $P(t)$. When $P(t)$ is a non-zero separable polynomial with degree at least 2, it is conjectured that the Hasse principle and weak approximation hold whenever the Brauer–Manin obstruction is empty. When K/k is cyclic and Schinzel’s Hypothesis is granted, work of Colliot-Thélène, Skorobogatov and Swinnerton-Dyer [11, Theorem 1.1] yields a positive answer to this question. Note that this result is already a special case of earlier work of Colliot-Thélène and Swinnerton-Dyer [10] on pencils of Severi–Brauer varieties, but this connection is only made clear in the discussion [11, page 10]. Note that when K/k is cyclic the Brauer–Manin obstruction is known to be empty if $P(t)$ is irreducible over k . (This follows from Corollary 2.6(c) of Colliot-Thélène, Harari and Skorobogatov [8], which shows that the Brauer group contains only vertical elements when K/k is cyclic. However it is not hard to show that the vertical part of the Brauer group is trivial when $P(t)$ is irreducible over k , using the remark on page 76 of [8].)

Finally we mention that there is potential for tackling the case in which $P(t)$ is an arbitrary polynomial which splits completely over k into d linear factors, at least in the case $k = \mathbb{Q}$, by using ideas from the work of Green and Tao [14] together with the main theorem from Green, Tao and Ziegler [15]. In this case the methods of Heath-Brown and Skorobogatov [18] reduce the problem (1.1) to one involving a system of equations

$$\mathbf{N}_{K/\mathbb{Q}}(\mathbf{x}_i) + a_i \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x}_0) = c_i \mathbf{N}_{K/\mathbb{Q}}(\mathbf{y}_i) \neq 0, \quad (1 \leq i \leq d-1).$$

In the language of [14], this is a system of linear forms of finite complexity. The machinery of Green and Tao, and of Green, Tao and Ziegler, allows one to handle such systems when the norms are replaced by primes, and it seems reasonable to hope that a variant of the method would allow one to handle the problem above. This plan was first mentioned to us by Professor Wooley.

It will be apparent from the foregoing survey that the most obvious open case is that in which $P(t)$ is an irreducible quadratic over k , and this is the goal of the present paper. Again we shall be dependent on techniques from analytic number theory which have not been fully developed for ground fields other than $k = \mathbb{Q}$, so we shall confine attention to this latter case. With this restriction our goal will be to establish the Hasse principle and weak approximation for

$$P(t) = \mathbf{N}_{K/\mathbb{Q}}(x_1, \dots, x_n) \neq 0, \tag{1.2}$$

under suitable assumptions on the extension K/\mathbb{Q} . Let $|\cdot|_\nu$ denote the ν -adic norm, which we extend to vectors by setting $|\mathbf{z}|_\nu := \max_{1 \leq i \leq m} |z_i|_\nu$, if $\mathbf{z} = (z_1, \dots, z_m)$. When $\nu = \infty$ we will simply write $|\cdot|_\infty = |\cdot|$. With this in mind the following is our main result.

Theorem 1. — *Let $P(t) \in \mathbb{Q}[t]$ be an irreducible quadratic polynomial and let K be a quartic extension of \mathbb{Q} containing a root of $P(t)$. Suppose that, for every $\nu \in \Omega_{\mathbb{Q}}$, we are given a solution $(t^{(\nu)}, \mathbf{x}^{(\nu)}) \in \mathbb{Q}_\nu^5$ of (1.2). Let $S \subset \Omega_{\mathbb{Q}}$ be any finite subset and let $\varepsilon > 0$. Then there is a solution $(t, \mathbf{x}) \in \mathbb{Q}^5$ of (1.2) such that*

$$|t - t^{(\nu)}|_\nu < \varepsilon, \quad |\mathbf{x} - \mathbf{x}^{(\nu)}|_\nu < \varepsilon, \tag{1.3}$$

for every $\nu \in S$. Thus the Hasse principle and weak approximation hold for (1.2).

It is interesting to note that our result is both unconditional and concerns field extensions K/\mathbb{Q} which may be non-cyclic. This marks a departure from the sort of results achieved in [11]. In fact our theorem answers in the affirmative a question posed by Colliot-Thélène, during the 2005 Bremen workshop “Rational points on curves — explicit methods”, about the Hasse principle for (1.2) in the special case that K is a biquadratic extension containing a root of $P(t)$ (cf. the questions at the close of §2 in the work of Colliot-Thélène, Harari and Skorobogatov [8]).

The proof of Theorem 1 relies on techniques from analytic number theory and is inspired by work of Fouvry and Iwaniec [12], who proved that there are infinitely many primes p of the form $a^2 + q^2$, with q also prime. More generally they showed how to produce primes of the form $a^2 + q^2$ with q from any sufficiently dense set. Our argument involves many complexities of detail, but also one major simplification, since we have only to produce integers in $N_{K/\mathbb{Q}}(K^\times)$, rather than primes.

We can generalise our result mildly, to include the case in which $P(t) = cQ(t)^u$ for an odd positive integer u , where $Q(t) \in \mathbb{Q}[t]$ is an irreducible quadratic polynomial. This is achieved by establishing a bijection between solutions of (1.2) and solutions of the corresponding equation in which $P(t)$ is quadratic. To do this we begin by choosing $e, f \in \mathbb{Z}$ for which $eu + 4f = 1$. The equation (1.2) becomes $cQ(t)^u = \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x})$, and raising to the power e we obtain $c^e Q(t)^{1-4f} = \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x})^e$, whence $c^e Q(t) = \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x})^e Q(t)^{4f}$. Since K has degree 4 over \mathbb{Q} , we deduce that $P_0(t) := c^e Q(t)$ is a norm from K whenever $P(t) = cQ(t)^u$ is a norm from K . The converse deduction is similar. Thus if $c^e Q(t) = \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x})$ then, raising both sides to the power u we find that $c^{1-4f} Q(t)^u = \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x})^u$, whence $cQ(t)^u = c^{4f} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x})^u$. Thus $P(t)$ is a norm from K whenever $P_0(t)$ is a norm from K . It is now immediate from Theorem 1 that we have the Hasse principle and weak approximation for $P(t) = cQ(t)^u$.

By a simple change of variable we may assume that $P(t) = c(1 - at^2)$ in Theorem 1, where c is a non-zero rational and a is a square-free integer. Let

$$L := \mathbb{Q}(\sqrt{a}).$$

The fields to which our theorem applies take the shape $K = L(\sqrt{\beta})$, with $\beta \in L$. In particular we have $L \subseteq K$ in the statement of Theorem 1. It turns out that most of our argument carries over to an arbitrary degree n extension K of \mathbb{Q} that contains L as a subfield. Given $P(t) = c(1 - at^2)$ as above, we suppose that $(t^{(\nu)}, \mathbf{x}^{(\nu)}) \in \mathbb{Q}_\nu^{n+1}$ are solutions of (1.2), for each $\nu \in \Omega_\mathbb{Q}$. Then we want to determine conditions on K , beyond the hypothesis $L \subseteq K$, such that for any finite set $S \subset \Omega_\mathbb{Q}$ we can find a solution $(t, \mathbf{x}) \in \mathbb{Q}^{n+1}$ of (1.2) for which the weak approximation condition (1.3) holds for each $\nu \in S$.

In pursuing this goal we may assume that $\{\omega_1, \dots, \omega_n\}$ is an integral basis for the ring of integers \mathfrak{o}_K , with $\omega_1 = 1$. By the transitivity of norms we have $N_{K/\mathbb{Q}} = N_{L/\mathbb{Q}} \circ N_{K/L}$, since $L \subseteq K$. Hence any norm from K to \mathbb{Q} is also a norm from L to \mathbb{Q} . We will make frequent use of this fact in our work.

Since $(1 - at^2) = N_{L/\mathbb{Q}}(1 + t\sqrt{a})$ it follows from the hypotheses of the theorem that the equation $c = N_{L/\mathbb{Q}}(u + v\sqrt{a})$ can be solved for $u, v \in \mathbb{Q}_\nu$ for any $\nu \in \Omega_\mathbb{Q}$. The Hasse norm principle therefore implies that there exists $\delta \in L^\times$ such that

$$c = N_{L/\mathbb{Q}}(\delta)^{-1}.$$

Thus it will suffice to work with the equation

$$1 - at^2 = N_{L/\mathbb{Q}}(\delta) \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x}) \neq 0, \quad (1.4)$$

rather than (1.2), with square-free $a \in \mathbb{Z}$ and non-zero $\delta \in L$. We are then given a finite set $S \subset \Omega_{\mathbb{Q}}$ and a solution $(t^{(\nu)}, \mathbf{x}^{(\nu)}) \in \mathbb{Q}_{\nu}^{n+1}$ of this equation for every $\nu \in \Omega_{\mathbb{Q}}$ and we wish to establish the existence of a solution $(t, \mathbf{x}) \in \mathbb{Q}^{n+1}$ such that (1.3) holds for every $\nu \in S$. Our plan is to achieve this by arranging that

$$\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w})(1 + t\sqrt{a}) = \delta \mathbf{N}_{K/L}(\mathbf{y}) \neq 0, \quad (1.5)$$

where

$$\mathbf{N}_{K/L}(y_1, \dots, y_n) := N_{K/L}(y_1\omega_1 + \dots + y_n\omega_n).$$

Then if $\beta_1 = y_1\omega_1 + \dots + y_n\omega_n$ and $\beta_2 = w_1\omega_1 + \dots + w_n\omega_n$ we obtain (1.4) on taking $x_1\omega_1 + \dots + x_n\omega_n$ to correspond to the element $\beta = \beta_1\beta_2^{-2}$. It will be convenient to write $\mathbf{x} = \mathbf{y} \cdot \mathbf{w}^{-2}$ for the vector \mathbf{x} produced by this construction.

In fact we will establish the following result, which demonstrates the Hasse principle and weak approximation for (1.5), for any number field K of degree n that contains L .

Theorem 2. — *Let $\delta \in L^{\times}$ and assume that $L \subseteq K$. Suppose that, for every $\nu \in \Omega_{\mathbb{Q}}$, we are given a solution $(t^{(\nu)}, \mathbf{w}^{(\nu)}, \mathbf{y}^{(\nu)}) \in \mathbb{Q}_{\nu}^{2n+1}$ of (1.5). Let $S \subset \Omega_{\mathbb{Q}}$ be any finite subset and let $\varepsilon > 0$. Then there is a solution $(t, \mathbf{w}, \mathbf{y}) \in \mathbb{Q}^{2n+1}$ of (1.5) such that*

$$|t - t^{(\nu)}|_{\nu} < \varepsilon, \quad |\mathbf{w} - \mathbf{w}^{(\nu)}|_{\nu} < \varepsilon, \quad |\mathbf{y} - \mathbf{y}^{(\nu)}|_{\nu} < \varepsilon,$$

for every $\nu \in S$.

In § 2 we will show how Theorem 1 follows from this result when K is a quadratic extension of L . This will be achieved via the following result.

Lemma 1. — *Let K be a quadratic extension of L . Let $S \subset \Omega_{\mathbb{Q}}$ be a finite set and let $\varepsilon > 0$ be given. Suppose that the equation*

$$c(1 - at^2) = \mathbf{N}_{K/\mathbb{Q}}(\mathbf{x}) \neq 0 \quad (1.6)$$

has solutions $(t^{(\nu)}, \mathbf{x}^{(\nu)})$ everywhere locally. Then there exists $\delta = \delta_{\varepsilon} \in L$ with $c = N_{L/\mathbb{Q}}(\delta)^{-1}$ such that

$$1 + t\sqrt{a} = \delta \mathbf{N}_{K/L}(\mathbf{x}) \neq 0 \quad (1.7)$$

has solutions $(t_0^{(\nu)}, \mathbf{x}_0^{(\nu)})$ everywhere locally, with

$$|t^{(\nu)} - t_0^{(\nu)}|_{\nu} < \varepsilon, \quad |\mathbf{x}^{(\nu)} - \mathbf{x}_0^{(\nu)}|_{\nu} < \varepsilon,$$

for every $\nu \in S$.

We should emphasise here that when we speak of local solutions we are thinking of zeros over \mathbb{Q}_{ν} of the polynomial, defined over \mathbb{Q}_{ν} , which specifies the equation. In particular, elements of the completions L_{μ} , for $\mu \nmid \nu$, do not occur.

It has been suggested to us by Professor Colliot-Thélène that the open descent method of Colliot-Thélène and Skorobogatov [7] might be used to establish a variant of Lemma 1 in which K is an arbitrary finite extension of L . The proposed lemma would then give the same conclusion as Lemma 1, but under the assumption that the solutions $(t^{(\nu)}, \mathbf{x}^{(\nu)})$ of (1.6) produce an adèlic point orthogonal to the Brauer group of the variety. Once combined with Theorem 2, this should demonstrate that the Brauer–Manin obstruction to the Hasse principle and to weak approximation is the only one for (1.2), when $P(t) \in \mathbb{Q}[t]$ is an irreducible quadratic polynomial and K is an arbitrary extension of \mathbb{Q} containing a root of $P(t)$. However

this would still leave open the difficult problem of calculating the Brauer group, which our route avoids.

By Lemma 1, given local solutions of (1.6), we may produce an equation (1.5) in which $\mathbf{w} = (1, 0, 0, \dots, 0)$ and $\mathbf{y} = \mathbf{x}$, and which has corresponding local solutions suitably close to those of (1.6). We may then use Theorem 2 to produce a global solution of (1.5) close to the given local solutions. Finally, taking the norm from L to \mathbb{Q} we obtain a suitable global solution of (1.6). It should be pointed out that this argument uses the fact that the map from (\mathbf{w}, \mathbf{y}) to $\mathbf{x} = \mathbf{y} \cdot \mathbf{w}^{-2}$ is continuous for $|\cdot|_\nu$ providing that we avoid a neighbourhood of $\mathbf{w} = \mathbf{0}$. We stress that the only point in the paper where we use our assumption that K/L is quadratic occurs in the proof of Lemma 1.

We proceed to indicate the initial steps in our treatment of (1.5) in Theorem 2. A suitable value of $t \in \mathbb{Q}$ will exist, providing that

$$\mathrm{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{y})) = 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) \neq 0. \quad (1.8)$$

Moreover, we will then have

$$t = \frac{1}{2} \mathrm{Tr}_{L/\mathbb{Q}} \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w})^{-1} \delta \mathbf{N}_{K/L}(\mathbf{y})}{\sqrt{a}} \right).$$

Thus if we have a solution of (1.8) in which \mathbf{y} and \mathbf{w} are sufficiently close to $\mathbf{y}^{(\nu)}$ and $\mathbf{w}^{(\nu)}$ it will be automatic that the corresponding solution t will be close to $t^{(\nu)}$. It follows that we have only to establish a suitable Hasse principle and weak approximation result for (1.8).

We must make a further manoeuvre before reaching our fundamental equation. As mentioned above, the work of Fouvry and Iwaniec handles primes. Using standard machinery, prime numbers are dealt with by means of “Type I sums” and “Type II sums”. Of these, Type II sums involve bilinear forms in which the prime number is replaced by a product of integers uv , which have to lie in suitable ranges. In our case we can insist that our norm $\mathbf{N}_{K/L}(\mathbf{y})$ is a product $\mathbf{N}_{K/L}(\mathbf{u})\mathbf{N}_{K/L}(\mathbf{v})$, thereby eliminating the need to consider Type I sums. Indeed, since we can specify the sizes of \mathbf{u} and \mathbf{v} , the treatment of the Type II sums will also be simplified somewhat. Thus instead of attacking (1.8) we shall consider the Diophantine equation

$$\mathrm{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{u})\mathbf{N}_{K/L}(\mathbf{v})) = 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) \neq 0,$$

with the aim of finding suitably localised solutions

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^n.$$

Let σ denote the non-trivial automorphism of L and suppose that $\{1, \tau\}$ is a \mathbb{Z} -basis for \mathfrak{o}_L , and hence also a \mathbb{Q} -basis for L . For technical reasons it will be convenient to replace the trace $\mathrm{Tr}_{L/\mathbb{Q}}$ by a “skew-trace”

$$\widetilde{\mathrm{Tr}}(x, y) := \mathrm{Tr}_{L/\mathbb{Q}}(xy^\sigma D_L^{-1})$$

for $x, y \in L$, where $D_L = \tau - \tau^\sigma$. Thus (D_L) is the different of L/\mathbb{Q} . On writing

$$x = \delta \mathbf{N}_{K/L}(\mathbf{u}), \quad y = (\mathbf{N}_{K/L}(\mathbf{v})D_L)^\sigma$$

our condition becomes

$$\widetilde{\mathrm{Tr}}(x, y) = 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) \neq 0. \quad (1.9)$$

We will count suitably restricted solutions of this equation. If \mathcal{N} is the number of such solutions we can write

$$\mathcal{N} = \sum_{x \in \mathfrak{o}_L} \sum_{y \in \mathfrak{o}_L} \alpha(x) \beta(y) \lambda(\widetilde{\text{Tr}}(x, y)).$$

Here the function $\alpha(x)$, respectively $\beta(y)$, counts appropriately restricted representations of x by $\delta \mathbf{N}_{K/L}(\mathbf{u})$, respectively of y by $(\mathbf{N}_{K/L}(\mathbf{v}) D_L)^\sigma$. Moreover $\lambda(l)$ counts suitably constrained solutions of $l = 2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w})$.

Our expression for \mathcal{N} can be viewed as a bilinear form. We have good techniques for estimating these, going back to the works of Vinogradov. However the methods are designed to produce upper bounds for bilinear forms in which we expect cancellation, while our problem is to establish an asymptotic formula for an expression in which all the terms are non-negative. We shall therefore split $\alpha(x)$ into two parts $\alpha(x) = \hat{\alpha}(x) + \alpha_0(x)$, and write

$$\mathcal{N} = \mathcal{M} + \mathcal{E},$$

where \mathcal{M} is a main term, and contains the contribution from $\hat{\alpha}(x)$, while \mathcal{E} is an error term, and is the corresponding expression involving $\alpha_0(x)$. Thus we will need $\hat{\alpha}(x)$ to be a sufficiently simple function that we can compute \mathcal{M} directly. Moreover we will want $\alpha_0(x)$ to produce sufficient cancellation on average, so that a bilinear form estimation of \mathcal{E} can be achieved. The underlying principle here is exactly that which Linnik [22] developed in his “dispersion method”.

In § 4 we will describe a general procedure for producing an approximation of the type $\hat{\alpha}(x)$. In our context x runs over the ring \mathfrak{o}_L , but we will begin by presenting the method as it applies to sequences indexed by \mathbb{Z} , since we hope this will prove to be of independent interest. With this choice made, our proof that the bilinear form \mathcal{E} makes a satisfactory overall contribution to the asymptotic formula is the subject of § 7. It is this part of our argument which is based on ideas from the work of Fouvry and Iwaniec [12], who provide a general framework for estimating sums of this sort. Finally, the asymptotic evaluation of \mathcal{M} will be executed in § 8 and § 9.

Acknowledgements. — Some of this work was done while the authors were visiting the *Hausdorff Institute* in Bonn and the *Institute for Advanced Study* in Princeton, and while the second author was visiting the *Mathematical Sciences Research Institute* in Berkeley. The hospitality and financial support of these bodies is gratefully acknowledged. While working on this paper the first author was supported by EPSRC grant number EP/E053262/1.

2. Deduction of Theorem 1

The goal of this section is to prove Lemma 1. As explained in the previous section, this is enough to allow the deduction of Theorem 1 from Theorem 2. Let $P(t) = c(1 - at^2)$, where c is a non-zero rational and a is a square-free integer. For the moment let K be an arbitrary number field of degree n containing $L = \mathbb{Q}(\sqrt{a})$. In particular n is even. Fix any $\delta_0 \in L^\times$ such that $c = N_{L/\mathbb{Q}}(\delta_0)^{-1}$ and let $S \subset \Omega_{\mathbb{Q}}$ be finite. We may assume that S contains the archimedean valuation, together with any valuations that become ramified in K , and any valuations $\nu \in \Omega_{\mathbb{Q}}$ for which $v_\mu(\delta_0) \neq 0$ for some $\mu \in \Omega_L$ above ν .

We proceed to look for a suitable δ , fulfilling the conditions of Lemma 1, by examining values $\delta = \delta_0 \gamma^\sigma \gamma^{-1}$. We will see that when $\nu \notin S$ any $\gamma \in L^\times$ is acceptable. Thus our first

task is to find a value γ which works for the “bad” places $\nu \in S$. For such valuations we claim that there exist $\gamma_1^{(\nu)}, \gamma_2^{(\nu)} \in \mathbb{Q}_\nu$, with

$$\gamma^{(\nu)} = \gamma_1^{(\nu)} + \gamma_2^{(\nu)} \sqrt{a} \neq 0,$$

such that

$$1 + t^{(\nu)} \sqrt{a} = \delta_0 \gamma^{(\nu)\sigma} \gamma^{(\nu)-1} \mathbf{N}_{K/L}(\mathbf{x}^{(\nu)}).$$

It then suffices to choose $\gamma \in L^\times$ so that γ is close to $\gamma^{(\nu)}$ for each $\nu \in S$, since then we may take $t_0^{(\nu)} = t^{(\nu)}$ and find a suitable $\mathbf{x}_0^{(\nu)}$ close to $\mathbf{x}^{(\nu)}$.

To establish the claim we begin by noting that the form $\mathbf{N}_{K/L}$ decomposes as

$$\mathbf{N}_{K/L} = N_1 + N_2 \sqrt{a}, \quad (2.1)$$

over L , where $N_1, N_2 \in \mathbb{Q}[x_1, \dots, x_n]$ are forms of degree $n/2$. Setting $\delta_0 = \delta_1 + \delta_2 \sqrt{a}$ and $\gamma^{(\nu)} = c_1 + c_2 \sqrt{a}$, and multiplying through by $\gamma^{(\nu)}$, we see that the equation becomes

$$\begin{aligned} (c_1 + c_2 \sqrt{a})(1 + t^{(\nu)} \sqrt{a}) \\ = (c_1 - c_2 \sqrt{a})(\delta_1 + \delta_2 \sqrt{a})(N_1(\mathbf{x}^{(\nu)}) + N_2(\mathbf{x}^{(\nu)}) \sqrt{a}). \end{aligned} \quad (2.2)$$

Thus our problem is to show the existence of $(c_1, c_2) \in \mathbb{Q}_\nu^2$ satisfying this, given the condition (1.4), namely

$$1 - at^{(\nu)2} = (\delta_1^2 - a\delta_2^2)(N_1(\mathbf{x}^{(\nu)})^2 - aN_2(\mathbf{x}^{(\nu)})^2) \neq 0. \quad (2.3)$$

If we set $A_1 = \delta_1 N_1(\mathbf{x}^{(\nu)}) + a\delta_2 N_2(\mathbf{x}^{(\nu)})$ and $A_2 = \delta_1 N_2(\mathbf{x}^{(\nu)}) + \delta_2 N_1(\mathbf{x}^{(\nu)})$ for convenience, then (2.2) becomes a pair of conditions

$$c_1(1 - A_1) + c_2(at^{(\nu)} + aA_2) = c_1(t^{(\nu)} - A_2) + c_2(A_1 + 1) = 0. \quad (2.4)$$

We need to find a solution c_1, c_2 of these, with $c_1^2 - ac_2^2 \neq 0$. In doing so we may assume (2.3), which becomes

$$1 - at^{(\nu)2} = A_1^2 - aA_2^2 \neq 0.$$

However the determinant of the system (2.4) is

$$\begin{aligned} (1 - A_1)(A_1 + 1) - (at^{(\nu)} + aA_2)(t^{(\nu)} - A_2) &= (1 - at^{(\nu)2}) - (A_1^2 - aA_2^2) \\ &= 0. \end{aligned}$$

Moreover, if $c_1 = \pm \sqrt{a}c_2 \neq 0$ one readily deduces from (2.2) that $1 - at^{(\nu)2} = 0$, which is impossible. This suffices for the proof of the claim.

In handling the case $\nu \notin S$ the following lemma will be useful. It will be proved at the end of the section.

Lemma 2. — *Let $\nu \in \Omega_{\mathbb{Q}}$ be a finite place, unramified in K . Suppose that*

$$\beta = b_1 + b_2 \sqrt{a} \in L^\times$$

is a unit in L_μ for each place μ of L above ν . Then $\beta = \mathbf{N}_{K/L}(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{Q}_\nu^n$, by which we mean that if N_1 and N_2 are as in (2.1) then $b_1 = N_1(\mathbf{x})$ and $b_2 = N_2(\mathbf{x})$.

While Lemma 2 is valid for arbitrary extensions K of L , we now make the assumption that K is a quadratic extension of L . In particular we have $n = 4$ in the above discussion.

When $\nu \notin S$ our task is to show that if

$$f(t) := (1 + t\sqrt{a})\gamma(\gamma^\sigma)^{-1}\delta_0^{-1},$$

then there exists $t_0^{(\nu)} \in \mathbb{Q}_\nu$ such that $f(t_0^{(\nu)})$ is of the form $\mathbf{N}_{K/L}(\mathbf{x}_0^{(\nu)})$ for some vector $\mathbf{x}_0^{(\nu)} \in \mathbb{Q}_\nu^4$. Since $\nu \notin S$ there are no weak approximation conditions to be satisfied. We begin by considering the case in which $\nu \notin S$ is inert in L/\mathbb{Q} . Then $v_\nu(\gamma^\sigma \gamma^{-1}) = 0$ for any γ , and $v_\nu(\delta_0) = 0$ by the choice of S , whence $\gamma(\gamma^\sigma)^{-1} \delta_0^{-1}$ will be a unit in L_ν . Lemma 2 then shows that there exists $\mathbf{x}_0^{(\nu)} \in \mathbb{Q}_\nu^4$ such that $f(0) = \mathbf{N}_{K/L}(\mathbf{x}_0^{(\nu)})$. It therefore suffices to take $t_0^{(\nu)} = 0$ in (1.7).

Finally we must deal with the case $\nu \notin S$, with ν split in L . Suppose ν splits as μ_1 and $\mu_2 = \mu_1^\sigma$ in L . Write p for the rational prime associated to ν . Let

$$v_{\mu_1}(\gamma) = e_1, \quad v_{\mu_2}(\gamma) = e_2.$$

Let \mathfrak{p}_1 and $\mathfrak{p}_2 = \mathfrak{p}_1^\sigma$ be the prime ideals associated to μ_1 and μ_2 respectively, so that $(\gamma) = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \mathfrak{g}$ for some ideal \mathfrak{g} coprime to both \mathfrak{p}_1 and \mathfrak{p}_2 . Choose $\alpha_1 \in \mathfrak{p}_1 \setminus (\mathfrak{p}_1^2 \cup \mathfrak{p}_2)$, and set $\alpha_2 = \alpha_1^\sigma$. We write $e = 2 + |e_1| + |e_2|$ and

$$h_1 = 1 + \frac{1}{2}(|e_1| + |e_2| + e_2 - e_1), \quad h_2 = 1 + \frac{1}{2}(|e_1| + |e_2| + e_1 - e_2),$$

so that e, h_1 and h_2 are integers, and e is strictly positive. Since $\alpha_i \in L$ and K/L is quadratic, we have $N_{K/L}(\alpha_i) = \alpha_i^2$. We may then calculate that

$$v_{\mu_1}(1 + p^{-e} \sqrt{a}) = -e, \quad v_{\mu_1}(\gamma) = e_1, \quad v_{\mu_1}(\gamma^\sigma) = e_2,$$

and

$$v_{\mu_1} \left(N_{K/L}(\alpha_1^{h_1} \alpha_2^{h_2}) \right) = 2h_1 v_{\mu_1}(\alpha_1) + 2h_2 v_{\mu_1}(\alpha_2) = 2h_1,$$

whence

$$v_{\mu_1} \left(f(p^{-e}) N_{K/L}(\alpha_1^{h_1} \alpha_2^{h_2}) \right) = -e + e_1 - e_2 + 2h_1 = 0.$$

Similarly we have

$$v_{\mu_2} \left(f(p^{-e}) N_{K/L}(\alpha_1^{h_1} \alpha_2^{h_2}) \right) = 0.$$

Thus Lemma 2 tells us that $f(p^{-e}) N_{K/L}(\alpha_1^{h_1} \alpha_2^{h_2})$ can be written as $\mathbf{N}_{K/L}(\mathbf{y})$ for some $\mathbf{y} \in \mathbb{Q}_\nu^4$. It follows that $f(p^{-e})$ takes the form $\mathbf{N}_{K/L}(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \mathbb{Q}_\nu^4$, whence $t_0^{(\nu)} = p^{-e}$ is acceptable in (1.7). This completes the proof of Lemma 1.

It now remains to establish Lemma 2, for which we return to an arbitrary extension K of degree n over \mathbb{Q} , which contains L as a subfield. It will be convenient to use the notation i_μ for the embedding of L into the completion L_μ , and similarly for valuations of \mathbb{Q} and K . Thus our hypothesis is that $i_\mu(\beta)$ is a unit for every $\mu \mid \nu$. For each such μ we choose a place $\lambda(\mu) \in \Omega_K$ such that $\lambda(\mu) \mid \mu$. Then the extension of local fields $K_{\lambda(\mu)}/L_\mu$ is unramified, whence $i_\mu(\beta)$ must be a norm from $K_{\lambda(\mu)}$ (see Gras [13, Corollary 1.4.3, part (ii), page 75], for example). For each μ above ν we may therefore write

$$i_\mu(\beta) = N_{K_{\lambda(\mu)}/L_\mu}(y_{\lambda(\mu)}),$$

for appropriate elements $y_{\lambda(\mu)} \in K_{\lambda(\mu)}$. If $\lambda \in \Omega_K$ lies above μ but is different from $\lambda(\mu)$ we take $y_\lambda = i_\lambda(1)$, so that

$$i_\mu(1) = N_{K_\lambda/L_\mu}(y_\lambda).$$

We now use weak approximation to find elements $y^{(j)} \in K$ such that

$$\left| i_\lambda(y^{(j)}) - y_\lambda \right|_\lambda < \frac{1}{j}$$

for every $\lambda \in \Omega_K$ above ν . We note in particular that the sequence $(y^{(j)})$ converges with respect to each valuation λ above ν . We now have

$$\lim_{j \rightarrow \infty} N_{K_\lambda/L_\mu}(i_\lambda(y^{(j)})) = \begin{cases} i_\mu(\beta), & \text{if } \lambda = \lambda(\mu) \text{ for some } \mu \mid \nu, \\ i_\mu(1), & \text{otherwise,} \end{cases}$$

where the limit is with respect to $|\cdot|_\mu$. We therefore conclude that

$$\prod_{\lambda \mid \mu} N_{K_\lambda/L_\mu}(i_\lambda(y^{(j)})) \rightarrow i_\mu(\beta).$$

However, according to Gras [13, Proposition 2.2, page 93] we have

$$\prod_{\lambda \mid \mu} N_{K_\lambda/L_\mu}(i_\lambda(y^{(j)})) = i_\mu(N_{K/L}(y^{(j)})),$$

so that

$$i_\mu(N_{K/L}(y^{(j)})) \rightarrow i_\mu(\beta).$$

Since this holds for all μ above ν it follows that

$$N_1(\mathbf{x}^{(j)}) \rightarrow b_1 \quad \text{and} \quad N_2(\mathbf{x}^{(j)}) \rightarrow b_2,$$

the convergence being with respect to $|\cdot|_\nu$, where

$$y^{(j)} = x_1^{(j)}\omega_1 + \cdots + x_n^{(j)}\omega_n.$$

Finally, since the sequence $y^{(j)} \in K$ converges for every valuation λ above ν , the sequence $\mathbf{x}^{(j)} \in \mathbb{Q}^n$ must converge in \mathbb{Q}_ν , yielding the required vector $\mathbf{x} \in \mathbb{Q}_\nu^n$.

3. Preliminaries for Theorem 2

We are now ready to begin the proof of Theorem 2, which concerns the Hasse principle and weak approximation for the variety (1.5). For the remainder of the paper let K be an arbitrary number field of degree n that contains $L = \mathbb{Q}(\sqrt{a})$. As noted, it will be convenient to work with the equivalent variety (1.8). For ease of reference we repeat the definition here:-

$$\mathrm{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{y})) = 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) \neq 0. \quad (3.1)$$

We are presented with local solutions $\mathbf{y}^{(\nu)}, \mathbf{w}^{(\nu)}$ for every valuation ν , and wish to find a global solution which approximates these for every $\nu \in S$. We claim that it suffices to consider the case in which $\mathbf{y}^{(\nu)}, \mathbf{w}^{(\nu)}$ are integral for every finite $\nu \in S$. Indeed, let us suppose that we can solve (3.1) with the local conditions

$$|\mathbf{y} - \mathbf{y}^{(\nu)}|_\nu < \varepsilon, \quad |\mathbf{w} - \mathbf{w}^{(\nu)}|_\nu < \varepsilon, \quad (3.2)$$

providing that $\mathbf{y}^{(\nu)}, \mathbf{w}^{(\nu)}$ are integral for all finite $\nu \in S$. Let us suppose further that we have a general set of values for $\mathbf{y}^{(\nu)}, \mathbf{w}^{(\nu)}$ satisfying (3.1). Then we may choose an integer $N \in \mathbb{N}$ such that $N^2 \mathbf{y}^{(\nu)}, N \mathbf{w}^{(\nu)}$ are integral for all finite $\nu \in S$. We note that these values will still satisfy (3.1). Then, by our assumption, we can find a global solution \mathbf{y}, \mathbf{w} of (3.1) which satisfies

$$|\mathbf{y} - N^2 \mathbf{y}^{(\nu)}|_\nu < \varepsilon |N^2|_\nu, \quad |\mathbf{w} - N \mathbf{w}^{(\nu)}|_\nu < \varepsilon |N|_\nu$$

for all $\nu \in S$. It follows that $N^{-2} \mathbf{y}, N^{-1} \mathbf{w}$ is a solution of (3.1) fulfilling the condition (3.2). This establishes our claim.

We now use weak approximation for \mathbb{Z}^n to produce vectors $\mathbf{y}^{(M)}, \mathbf{w}^{(M)} \in \mathbb{Z}^n$ such that

$$|\mathbf{y}^{(M)} - \mathbf{y}^{(\nu)}|_\nu < \varepsilon, \quad |\mathbf{w}^{(M)} - \mathbf{w}^{(\nu)}|_\nu < \varepsilon$$

for all finite $\nu \in S$. Thus (3.2) becomes

$$\mathbf{y} \equiv \mathbf{y}^{(M)} \pmod{M}, \quad \mathbf{w} \equiv \mathbf{w}^{(M)} \pmod{M} \quad (3.3)$$

for an appropriate modulus $M \in \mathbb{N}$.

Having suitably re-interpreted the weak approximation conditions for the finite places we turn our attention to the infinite place. Here we use a similar re-scaling argument to conclude that if $Y, W \in \mathbb{N}$ satisfy

$$W \equiv 1 \pmod{M} \quad \text{and} \quad Y = W^2,$$

then a solution \mathbf{y}, \mathbf{w} of (3.1) which satisfies both (3.3) and the $\nu = \infty$ constraints

$$|\mathbf{y} - Y\mathbf{y}^{(\mathbb{R})}| < \varepsilon Y, \quad |\mathbf{w} - W\mathbf{w}^{(\mathbb{R})}| < \varepsilon W, \quad (3.4)$$

gives rise to a solution $Y^{-1}\mathbf{y}, W^{-1}\mathbf{w}$ of (3.1) which meets the original condition (3.2). Since $\mathbf{y}^{(\mathbb{R})}$ and $\mathbf{w}^{(\mathbb{R})}$ cannot vanish we may choose ε sufficiently small that neither of \mathbf{y} or \mathbf{w} can be zero in (3.4).

As in § 1 we replace the vector \mathbf{y} by \mathbf{u} and \mathbf{v} to produce the variety (1.9), whose definition it is convenient to repeat here:-

$$\mathcal{J} : \widetilde{\text{Tr}}(\delta \mathbf{N}_{K/L}(\mathbf{u}), (\mathbf{N}_{K/L}(\mathbf{v})D_L)^\sigma) = 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) \neq 0. \quad (3.5)$$

It is clear that if \mathbf{u} and \mathbf{v} are sufficiently close to $\mathbf{u}^{(\nu)} := \mathbf{y}^{(\nu)}$ and

$$\mathbf{v}^{(\nu)} := (1, 0, 0, \dots, 0) \quad (3.6)$$

in \mathbb{Q}_ν then \mathbf{y} will be suitably close to $\mathbf{y}^{(\nu)}$. We therefore assume that (3.5) has local solutions $\mathbf{u}^{(\nu)}, \mathbf{v}^{(\nu)}, \mathbf{w}^{(\nu)}$ for all places ν , with $\mathbf{v}^{(\nu)}$ given by (3.6), and we aim to find a global solution such that

$$|\mathbf{u} - \mathbf{u}^{(\nu)}|_\nu < \varepsilon, \quad |\mathbf{v} - \mathbf{v}^{(\nu)}|_\nu < \varepsilon, \quad |\mathbf{w} - \mathbf{w}^{(\nu)}|_\nu < \varepsilon, \quad (3.7)$$

for $\nu \in S$. Since $\mathbf{u}^{(\nu)}$ and $\mathbf{v}^{(\nu)}$ are integral at all finite $\nu \in S$ we can re-interpret the corresponding conditions as congruences

$$\mathbf{u} \equiv \mathbf{u}^{(M)} \pmod{M}, \quad \mathbf{v} \equiv \mathbf{v}^{(M)} \pmod{M}$$

with integer vectors $\mathbf{u}^{(M)}$ and $\mathbf{v}^{(M)}$ for which

$$\mathbf{v}^{(M)} := (1, 0, 0, \dots, 0). \quad (3.8)$$

For technical reasons we will move $\mathbf{u}^{(\mathbb{R})}$ in (3.5) very slightly, and make a corresponding adjustment in $\mathbf{w}^{(\mathbb{R})}$ to compensate, so as to ensure that

$$\frac{\partial \mathbf{N}_{K/\mathbb{Q}}(\mathbf{u}^{(\mathbb{R})})}{\partial u_i} \neq 0, \quad (1 \leq i \leq n). \quad (3.9)$$

For the infinite place we replace the parameter Y by two further values U and V satisfying $UV = Y$ and impose the conditions

$$|\mathbf{u} - U\mathbf{u}^{(\mathbb{R})}| < \varepsilon U, \quad |\mathbf{v} - V\mathbf{v}^{(\mathbb{R})}| < \varepsilon V$$

instead of $|\mathbf{y} - Y\mathbf{y}^{(\mathbb{R})}| < \varepsilon Y$.

We can now summarise our conclusions in the following result.

Lemma 3. — Suppose we are given local solutions of (3.5) for every valuation ν of \mathbb{Q} , subject to the condition (3.6). Let $\varepsilon > 0$ also be given. Then there is a modulus $M \in \mathbb{N}$ having $|M|_\nu < 1$ for all finite $\nu \in S$, and a solution $(\mathbf{u}^{(M)}, \mathbf{v}^{(M)}, \mathbf{w}^{(M)})$ of (3.5) over $\mathbb{Z}/M\mathbb{Z}$ satisfying (3.8), having the following property. Let V, H_0 be integer parameters with $H_0 \equiv V \equiv 1 \pmod{M}$. Let $H = H_0^2$ and suppose that $V \geq H \geq 1$. If

$$U = HV, \quad W = H^{1/2}V,$$

then any solution $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{Z}^{3n}$ of (3.5) satisfying

$$\begin{aligned} \mathbf{u} &\equiv \mathbf{u}^{(M)} \pmod{M}, \\ \mathbf{v} &\equiv \mathbf{v}^{(M)} \pmod{M}, \\ \mathbf{w} &\equiv \mathbf{w}^{(M)} \pmod{M}, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |\mathbf{u} - U\mathbf{u}^{(\mathbb{R})}| &< \varepsilon U, \\ |\mathbf{v} - V\mathbf{v}^{(\mathbb{R})}| &< \varepsilon V, \\ |\mathbf{w} - W\mathbf{w}^{(\mathbb{R})}| &< \varepsilon W, \end{aligned} \tag{3.11}$$

gives rise to a global solution of (3.5) satisfying (3.7). Moreover, for every finite place $\nu \in S$ there is a solution $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{Z}_\nu^{3n}$ of (3.5) satisfying (3.10).

It might help the reader at this point to say more about the rôle of the parameters H and V . We shall think of H as being a small fixed power of V . When we estimate error terms in our analysis we cannot afford to lose any power V^θ of V , unless θ can be taken arbitrarily small. On the other hand there will be certain points in our argument where we will lose factors of V^η with arbitrary small $\eta > 0$. This will not matter since we will make a key saving which is a power of H , so that there is a net gain overall.

With this in mind, many of our estimates will involve factors of the type $V^{O(\eta)}$. These involve the standard convention that there are implicit order constants for each occurrence of the $O(\cdot)$ notation, which need not be the same on each occasion. Since we are taking the degree n of K to be fixed, we will allow these implicit order constants to depend on n . Recalling that $H \leq V$ we may replace terms involving any combinations of H^η, U^η and W^η by $V^{O(\eta)}$. The number η will be a sufficiently small positive constant, which will be fixed throughout the proof. We could have chosen to specify its value at the outset of the argument, but we feel it is more instructive merely to impose the condition that η is sufficiently small, at various points in the proof.

We are now ready to cast our problem in terms of a bilinear form. If R is any ring it will be convenient to write $\mathcal{J}(R)$ for the set of solutions $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of the equation (3.5) in which each of the vectors has coordinates lying in R . In the light of Lemma 3 we introduce the counting function

$$N(H, V) := \#\{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{J}(\mathbb{Z}) : (3.10) \text{ and } (3.11) \text{ hold}\}.$$

Henceforth we will allow the constants implied by the notations \ll , \gg and $O(\cdot)$ to depend on

$$\mathbf{u}^{(\mathbb{R})}, \mathbf{v}^{(\mathbb{R})}, \mathbf{w}^{(\mathbb{R})}, \mathbf{u}^{(M)}, \mathbf{v}^{(M)}, \mathbf{w}^{(M)}, M, \delta, L, K, \text{ and } \varepsilon,$$

which are to be regarded as fixed once and for all.

In our work we will restrict the values over which \mathbf{v} runs by stipulating that if $\mathbf{N}_{K/L}(\mathbf{v}) = \mathbf{N}_1(\mathbf{v}) + \mathbf{N}_2(\mathbf{v})\tau$ then

$$\gcd(\mathbf{N}_1(\mathbf{v}), \mathbf{N}_2(\mathbf{v})) = 1. \quad (3.12)$$

In particular it follows that

$$\gcd(a, b) = O(1) \quad \text{for} \quad (\mathbf{N}_{K/L}(\mathbf{v})D_L)^\sigma = a + b\tau. \quad (3.13)$$

We are now ready to specify the sets over which we will sum. In the case of the variable \mathbf{u} there is a technical point to be dealt with in § 5. For the time being we give ourselves independent linear forms $L_1(\mathbf{u}), \dots, L_n(\mathbf{u})$ whose rôle will become clear later. Let G be a further parameter, tending to infinity with V , which we assume is in the range $1 \leq G \leq H$. We then define the regions

$$\begin{aligned} \mathcal{U} &:= \left\{ \mathbf{u} \in \mathbb{R}^n : \max_{1 \leq i \leq n} |L_i(\mathbf{u}) - UL_i(\mathbf{u}^{(\mathbb{R})})| < G^{-1}U \right\}, \\ \mathcal{V} &:= \left\{ \mathbf{v} \in \mathbb{R}^n : |\mathbf{v} - V\mathbf{v}^{(\mathbb{R})}| < G^{-1}V \right\}, \\ \mathcal{W} &:= \left\{ \mathbf{w} \in \mathbb{R}^n : |\mathbf{w} - W\mathbf{w}^{(\mathbb{R})}| < G^{-1}W \right\}. \end{aligned} \quad (3.14)$$

In order to interpret our counting function $N(H, V)$ as a bilinear form, we let

$$\alpha(x) := \#\{\mathbf{u} \in \mathcal{U} \cap \mathbb{Z}^n : \mathbf{u} \equiv \mathbf{u}^{(M)} \pmod{M}, \delta\mathbf{N}_{K/L}(\mathbf{u}) = x\} \quad (3.15)$$

and

$$\beta(y) := \#\left\{ \mathbf{v} \in \mathcal{V} \cap \mathbb{Z}^n : \begin{array}{l} \mathbf{v} \equiv \mathbf{v}^{(M)} \pmod{M}, \\ (3.12) \text{ holds and } (\mathbf{N}_{K/L}(\mathbf{v})D_L)^\sigma = y \end{array} \right\}, \quad (3.16)$$

for $x, y \in \mathfrak{o}_L$. Lastly we define the function

$$\lambda(l) := \#\left\{ \mathbf{w} \in \mathcal{W} \cap \mathbb{Z}^n : \mathbf{w} \equiv \mathbf{w}^{(M)} \pmod{M}, 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) = l \right\}, \quad (3.17)$$

on \mathbb{Z} . Notice that $\mathbf{u}^{(\mathbb{R})}, \mathbf{v}^{(\mathbb{R})}$ and $\mathbf{w}^{(\mathbb{R})}$ are all non-zero, whence \mathbf{u}, \mathbf{v} and \mathbf{w} will be non-zero throughout \mathcal{U}, \mathcal{V} and \mathcal{W} , if G is large enough. It follows in particular that α, β and λ are supported on non-zero $x, y \in \mathfrak{o}_L$ and $l \in \mathbb{Z}$.

We now define the bilinear form

$$\mathcal{N}(G, H, V) := \sum_{x \in \mathfrak{o}_L} \sum_{y \in \mathfrak{o}_L} \alpha(x) \beta(y) \lambda(\widetilde{\text{Tr}}(x, y)).$$

It is easy to check that $N(H, V) \geq \mathcal{N}(G, H, V)$ whenever $G \gg \varepsilon^{-1}$, where the implied constant is allowed to depend on the linear forms L_1, \dots, L_n , a convention that we adhere to for the remainder of the paper. It now suffices to demonstrate that $\mathcal{N}(G, H, V) > 0$ for large values of G . Note that we will ultimately take $G = \log V$.

Although we have framed $\mathcal{N}(G, H, V)$ as a bilinear form, it is not an upper bound for $\mathcal{N}(G, H, V)$ that we seek but an asymptotic formula as $G \rightarrow \infty$. As indicated in the introduction we will begin by extracting a main term from our expression for $\mathcal{N}(G, H, V)$. Instrumental in this will be finding a decomposition $\alpha(x) = \hat{\alpha}(x) + \alpha_0(x)$, for an appropriate approximation $\hat{\alpha}(x)$ to $\alpha(x)$. We will then write

$$\mathcal{N}(G, H, V) = \mathcal{M}(G, H, V) + \mathcal{E}(G, H, V), \quad (3.18)$$

where

$$\mathcal{M}(G, H, V) := \sum_{x \in \mathfrak{o}_L} \sum_{y \in \mathfrak{o}_L} \hat{\alpha}(x) \beta(y) \lambda(\widetilde{\text{Tr}}(x, y))$$

is regarded as the main term and

$$\mathcal{E}(G, H, V) := \sum_{x \in \mathfrak{o}_L} \sum_{y \in \mathfrak{o}_L} \alpha_0(x) \beta(y) \lambda(\widetilde{\text{Tr}}(x, y))$$

is the error term. The handling of $\mathcal{E}(G, H, V)$ will be executed in § 7 and the estimation of $\mathcal{M}(G, H, V)$ will be the subject of § 8 and § 9.

Our treatment of $\mathcal{E}(G, H, V)$ requires bounds for

$$\sum_{\substack{x \in R \\ x \equiv x_0 \pmod{h}}} \alpha_0(x)$$

uniformly for small moduli h and square regions R . Thus our approximation $\hat{\alpha}$ will have to be such that α_0 averages to zero over all congruence classes to small moduli. This will be achieved via a quite general procedure described in the next section. Our estimate for $\mathcal{E}(G, H, V)$ will also require bounds for

$$\sum_{x \in \mathfrak{o}_L} |\alpha_0(x)|^2, \quad \sum_{y \in \mathfrak{o}_L} |\beta(y)|^2 \quad \text{and} \quad \sum_{l \in \mathbb{Z}} |\lambda(l)|^2,$$

for which we have the following result.

Lemma 4. — *For any $\eta > 0$ we have*

$$\sum_{x \in \mathfrak{o}_L} |\alpha(x)|^2 \ll_{\eta} U^{n+\eta}, \quad \sum_{y \in \mathfrak{o}_L} |\beta(y)|^2 \ll_{\eta} V^{n+\eta},$$

and

$$\sum_{l \in \mathbb{Z}} |\lambda(l)|^2 \ll_{\eta} W^{n+\eta}.$$

Since $|\alpha_0(x)|^2 \leq 2|\alpha(x)|^2 + 2|\hat{\alpha}(x)|^2$ we will also require a bound for

$$\sum_{x \in \mathfrak{o}_L} |\hat{\alpha}(x)|^2.$$

This will be established in § 5. We conclude this section by proving Lemma 4. We will discuss the upper bound for the case of β , the remaining estimates being dealt with similarly. Let $h(\mathbf{v}) = (\mathbf{N}_{K/L}(\mathbf{v})D_L)^{\sigma}$. We shall show that if \mathbf{v}' is given, then there are $O_{\eta}(V^{\eta})$ choices of \mathbf{v} for which $h(\mathbf{v}) = h(\mathbf{v}')$. We set $\varpi = h(\mathbf{v}')$, so that $\varpi \in \mathfrak{o}_L \setminus \{0\}$. If $\rho = v_1\omega_1 + \cdots + v_n\omega_n \in \mathfrak{o}_K$, then $\rho \mid \varpi$ in \mathfrak{o}_K , and the conjugates of ρ are all $O(V)$ in absolute magnitude. The number of admissible elements ρ is therefore

$$\ll_{\xi} V^{\xi} |N_{L/\mathbb{Q}}(\varpi)|^{\xi} \ll_{\xi} V^{\xi} (V^n)^{\xi} = V^{(1+n)\xi}$$

for any $\xi > 0$. If we now take $\xi = \eta/(1+n)$ it follows that the number of \mathbf{v} corresponding to a given \mathbf{v}' is $O_{\eta}(V^{\eta})$, as required.

The fact that a given value $\varpi = h(\mathbf{v})$ is attained $O(V^{\eta})$ times will be used at various points in the rest of the paper without further comment. Similarly, we shall use estimates $O(V^{\eta})$ for the number of representations of ϖ as $\mathbf{N}_{K/L}(\mathbf{u})$ with $\mathbf{u} \in \mathcal{U}$, or of l as $2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w})$ with $\mathbf{w} \in \mathcal{W}$.

4. A general approximation principle

Our goal now is to split $\alpha(x)$ into two parts $\alpha(x) = \hat{\alpha}(x) + \alpha_0(x)$, where $\hat{\alpha}(x)$ is a sufficiently simple function that we can compute $\mathcal{M}(G, H, V)$ directly. Moreover we will want $\alpha_0(x)$ to produce sufficient cancellation on average, so that a bilinear form estimation of $\mathcal{E}(G, H, V)$ can be achieved.

We start by describing a general procedure for producing an approximation of the type $\hat{\alpha}(x)$. In our context x runs over the ring \mathfrak{o}_L , but we will begin by presenting the method as it applies to sequences indexed by \mathbb{Z} , since we hope this will prove to be of independent interest. The underlying ideas are perhaps not new. In particular there are certain similar features in recent work of Brüdern [3]. However we have not found anything in the literature which exactly meets our needs.

We begin by supposing that we are given a sequence $k(1), \dots, k(N)$ of complex numbers. We aim to show how to approximate $k(n)$ locally by a function $\hat{k}(n)$. By this we mean that for any congruence class $a \pmod{q}$ the sum

$$S(a, q) := \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} k(n)$$

will be approximated by

$$\hat{S}(a, q) := \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \hat{k}(n),$$

at least for small values of q . Naturally we can do this by merely setting $\hat{k}(n) = k(n)$, but we seek a function $\hat{k}(n)$ which is defined in terms of the density of the sequence $k(n)$ in congruence classes to small moduli.

As an example of what we have in mind, consider the sequence $k(n) = \Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function. If we choose any constant $A \geq 3$ and set

$$\Lambda_Q(n) := \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \pmod{q} \\ \gcd(a, q) = 1}} e_q(an),$$

with $Q = (\log x)^A$, then

$$\sum_{\substack{n \leq x \\ n \equiv b \pmod{h}}} \Lambda(n) = \sum_{\substack{n \leq x \\ n \equiv b \pmod{h}}} \Lambda_Q(n) + O_A(x(\log x)^{-A/2}) \quad (4.1)$$

uniformly for $h \leq Q$, for all residue classes $b \pmod{h}$. This result follows from Heath-Brown [17, Lemma 1].

We first introduce our fundamental hypotheses. We assume that we have an arithmetic function $\rho(a, q)$ and a “smooth” function $\omega(n)$, together with a bound $E \geq 1$ such that

$$|S(a, q) - \rho(a, q)S| \leq E, \quad (4.2)$$

with

$$S = \sum_{n \leq N} \omega(n),$$

for all residue classes a to moduli $q \leq Q$. We observe that

$$\sum_{\substack{a \pmod{rs} \\ a \equiv b \pmod{r}}} S(a, rs) = S(b, r)$$

for all b, r, s , and we therefore impose the natural condition that

$$\sum_{\substack{a \pmod{rs} \\ a \equiv b \pmod{r}}} \rho(a, rs) = \rho(b, r) \quad (4.3)$$

for all b, r, s . Since we can always re-scale the functions ρ and ω in (4.2) there is no loss in generality in assuming that

$$\rho(0, 1) = 1. \quad (4.4)$$

Although it is not necessary in general, it will prove convenient to assume that

$$\rho(a, q) \in \mathbb{R}, \quad \rho(a, q) \geq 0$$

for all pairs a, q .

We will also require a smoothness condition on the function $\omega(n)$, which we formulate as the bound

$$\left| \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \omega(n) - q^{-1} S \right| \leq W, \quad (4.5)$$

for all residue classes a to moduli $q \leq Q^2$.

Our choice for $\hat{k}(n)$ will be motivated by the treatment of major arcs in the circle method. If we consider the exponential sum

$$\Sigma(\alpha) := \sum_{n \leq N} k(n) e(\alpha n),$$

then when α is close to a/q one would use the major-arc approximation

$$\left\{ \sum_{c \pmod{q}} \rho(c, q) e_q(ac) \right\} \left\{ \sum_{n \leq N} \omega(n) e((\alpha - a/q)n) \right\}.$$

When α is not close to a/q the above expression tends to be small. Hence it is reasonable to approximate $\Sigma(\alpha)$ by

$$\sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ \gcd(a, q) = 1}} \left\{ \sum_{c \pmod{q}} \rho(c, q) e_q(ac) \right\} \left\{ \sum_{n \leq N} \omega(n) e((\alpha - a/q)n) \right\}$$

for all real α . Picking out the coefficient of $e(\alpha n)$ in the above expression we are therefore led to suggest the choice

$$\hat{k}(n) := \omega(n) \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ \gcd(a, q) = 1}} e_q(-an) \sum_{c \pmod{q}} \rho(c, q) e_q(ac). \quad (4.6)$$

We proceed to investigate the sum $\hat{S}(b, h)$ with $h \leq Q$. We have

$$\hat{S}(b, h) = \sum_{\substack{n \leq N \\ n \equiv b \pmod{h}}} \omega(n) \sum_{q \leq Q} \sum_{c \pmod{q}} \rho(c, q) \sum_{\substack{a \pmod{q} \\ \gcd(a, q) = 1}} e_q(a(c - n)).$$

The final sum over $a \pmod{q}$ is a Ramanujan sum, for which the standard evaluation as

$$\sum_{d \mid \gcd(q, c-n)} d\mu(q/d)$$

produces

$$\hat{S}(b, h) = \sum_{q \leq Q} \sum_{c \pmod{q}} \rho(c, q) \sum_{d \mid q} d\mu(q/d) \sum_{\substack{n \leq N \\ n \equiv b \pmod{h} \\ n \equiv c \pmod{d}}} \omega(n).$$

The simultaneous congruences $n \equiv b \pmod{h}$ and $n \equiv c \pmod{d}$ are only soluble if $\gcd(d, h)$ divides $c - b$, in which case there is a unique solution $e \pmod{[d, h]}$ say. Thus if $\gcd(d, h) \mid c - b$ our hypothesis (4.5) shows that

$$\hat{S}(b, h) = \mathcal{M} + \mathcal{E},$$

with

$$\mathcal{M} = S \sum_{q \leq Q} \sum_{c \pmod{q}} \rho(c, q) \sum_{\substack{d \mid q \\ \gcd(d, h) \mid c-b}} \frac{d\mu(q/d)}{[d, h]}$$

and

$$|\mathcal{E}| \leq W \sum_{q \leq Q} \sum_{c \pmod{q}} \rho(c, q) \sum_{d \mid q} d|\mu(q/d)|.$$

In view of (4.3) and (4.4) we have

$$\sum_{c \pmod{q}} \rho(c, q) = \rho(0, 1) = 1,$$

whence the crude bound

$$\sum_{d \mid q} d|\mu(q/d)| \leq q^2$$

yields $|\mathcal{E}| \leq WQ^3$.

Turning to the main term \mathcal{M} we observe in general that

$$\begin{aligned} \sum_{\substack{d \mid q \\ \gcd(d, h) \mid k}} \frac{d\mu(q/d)}{[d, h]} &= h^{-1} \sum_{\substack{d \mid q \\ \gcd(d, h) \mid k}} \mu(q/d)(d, h) \\ &= h^{-1} \sum_{d \mid q} \mu(q/d) \sum_{e \mid d, h, k} e \sum_{f \mid d/e, h/e} \mu(f) \\ &= h^{-1} \sum_{e \mid q, h, k} e \sum_{f \mid q/e, h/e} \mu(f) \sum_{\substack{d \mid q \\ ef \mid d}} \mu(q/d) \\ &= h^{-1} \sum_{e \mid q, h, k} e \sum_{f \mid q/e, h/e} \mu(f) \sum_{g \mid q/(ef)} \mu\left(\frac{q/(ef)}{g}\right). \end{aligned}$$

The final sum vanishes unless $ef = q$, in which case we must have $q \mid h$. It then follows that

$$\begin{aligned} \sum_{\substack{d \mid q \\ \gcd(d, h) \mid k}} \frac{d\mu(q/d)}{[d, h]} &= h^{-1} \sum_{e \mid q, k} e\mu(q/e) \\ &= h^{-1} \sum_{\substack{a \pmod{q} \\ \gcd(a, q) = 1}} e_q(ak). \end{aligned}$$

Inserting this result into our formula for the main term \mathcal{M} , we see that

$$\mathcal{M} = Sh^{-1} \sum_{q \mid h} \sum_{c \pmod{q}} \rho(c, q) \sum_{\substack{a \pmod{q} \\ \gcd(a, q) = 1}} e_q(a(c - b)).$$

Using (4.3) we have

$$\rho(c, q) = \sum_{\substack{d \pmod{h} \\ d \equiv c \pmod{q}}} \rho(d, h),$$

whence

$$\mathcal{M} = Sh^{-1} \sum_{d \pmod{h}} \rho(d, h) \sum_{q \mid h} \sum_{\substack{a \pmod{q} \\ \gcd(a, q) = 1}} e_q(a(d - b)).$$

As q runs over divisors of h , and a runs over residue classes coprime to q , the fractions a/q run over the entire set

$$\left\{ \frac{n}{h} : 0 \leq n < h \right\}.$$

We therefore deduce that

$$\mathcal{M} = Sh^{-1} \sum_{d \pmod{h}} \rho(d, h) \sum_{n \pmod{h}} e_h(n(d - b)) = S\rho(b, h),$$

since the summation over n produces the value h when $d \equiv b \pmod{h}$, and the value 0 otherwise.

In view of (4.2) we may now summarise our results as follows.

Lemma 5. — *With the above assumptions we have*

$$|S(b, h) - \hat{S}(b, h)| \leq WQ^3 + E$$

for all $h \leq Q$ and all residue classes b modulo h .

Thus $\hat{k}(n)$ approximates $k(n)$ well, in congruence classes to small moduli. It may be instructive to consider the effect of this procedure on the sequence $k(n) = \Lambda(n)$ that we discussed earlier. Taking

$$\rho(a, q) = \begin{cases} 1/\varphi(q), & \text{if } \gcd(a, q) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

we readily deduce from the Siegel–Walfisz theorem that for any $A \geq 1$ there exists a constant $C_A > 0$ such that

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \Lambda(n) = \rho(a, q)N + O\left(N \exp(-C_A(\log N)^{1/2})\right),$$

uniformly for $q \leq (\log N)^A$. On taking $\omega(n) = 1$ we see that $E \ll N \exp(-C_A(\log N)^{1/2})$ is admissible in (4.2). Since $W \ll 1$ in (4.5), the approximation in (4.1) is a trivial consequence of Lemma 5 with

$$\begin{aligned} \Lambda_Q(n) &= \omega(n) \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ \gcd(a,q)=1}} e_q(-an) \sum_{\substack{b \pmod{q} \\ \gcd(b,q)=1}} \frac{e_q(ab)}{\varphi(q)} \\ &= \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \pmod{q} \\ \gcd(a,q)=1}} e_q(an), \end{aligned}$$

for $q \leq (\log N)^A$.

In applications it may be important to know about the size of $\hat{k}(n)$, and we therefore investigate the mean square

$$\Sigma := \sum_{n \leq N} |\hat{k}(n)|^2.$$

If we write

$$c_{a,q} := \sum_{b \pmod{q}} \rho(b,q) e_q(ab)$$

then

$$\hat{k}(n) = \omega(n) \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ \gcd(a,q)=1}} c_{a,q} e_q(-an).$$

Thus, if we assume that

$$|\omega(n)| \leq \omega_0$$

for all $n \in \mathbb{N}$, then the dual large sieve produces

$$\begin{aligned} \Sigma &\leq \omega_0^2 \sum_{n \leq N} \left| \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ \gcd(a,q)=1}} c_{a,q} e_q(-an) \right|^2 \\ &\leq \omega_0^2 (N + Q^2) \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ \gcd(a,q)=1}} |c_{a,q}|^2. \end{aligned}$$

In view of (4.2) we have

$$|Sc_{a,q} - T(a,q)| \leq qE,$$

with

$$T(a,q) := \sum_{b \pmod{q}} S(b,q) e_q(ab) = \sum_{n \leq N} k(n) e_q(an).$$

It follows that $|Sc_{a,q}|^2 \leq 2|T(a,q)|^2 + 2q^2 E^2$, and hence that

$$|S|^2 \Sigma \leq 2\omega_0^2 (N + Q^2) \left(E^2 Q^4 + \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ \gcd(a,q)=1}} |T(a,q)|^2 \right).$$

We now apply the standard large sieve to deduce that

$$|S|^2 \Sigma \leq 2\omega_0^2(N+Q^2) \left(E^2 Q^4 + (N+Q^2) \sum_{n \leq N} |k(n)|^2 \right).$$

We express this result formally in the following lemma.

Lemma 6. — *We have*

$$\sum_{n \leq N} |\hat{k}(n)|^2 \leq 2|S|^{-2} \omega_0^2(N+Q^2) \left(E^2 Q^4 + (N+Q^2) \sum_{n \leq N} |k(n)|^2 \right).$$

Moreover

$$\sum_{n \leq N} |\hat{k}(n)|^2 \ll \left(\frac{\omega_0 N}{|S|} \right)^2 \left(N + \sum_{n \leq N} |k(n)|^2 \right),$$

providing that $EQ^2 \leq N$.

Thus under suitable circumstances the L^2 -norm of $\hat{k}(n)$ will have order of magnitude bounded by the L^2 -norm of $k(n)$.

Having described the situation for sequences $k(n)$ indexed by \mathbb{Z} we return to our original problem, in which we have a sequence $\alpha(x)$ with x running over \mathfrak{o}_L . We will describe the situation as it applies to a quite general function α , and only later, in § 5, restrict to the function (3.15). We shall consider sums in which x runs over a region R , say, and lies in a congruence class $y \pmod{q}$, where $y \in \mathfrak{o}_L$ and $q \in \mathbb{N}$. We therefore set

$$S(y, q) := \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha(x) \tag{4.7}$$

and

$$\hat{S}(y, q) := \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \hat{\alpha}(x),$$

and assume that we have functions $\rho(y, q)$ and $\omega(x)$ such that $\rho(y, q)$ is non-negative and

$$|S(y, q) - \rho(y, q)S| \leq E \tag{4.8}$$

for $q \leq Q$, with $E \geq 1$ and

$$S := \sum_{x \in R} \omega(x).$$

As before we will assume that

$$\sum_{\substack{y \pmod{rs} \\ y \equiv z \pmod{r}}} \rho(y, rs) = \rho(z, r) \tag{4.9}$$

and

$$\rho(0, 1) = 1. \tag{4.10}$$

Finally, we shall suppose that

$$\left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \omega(x) - q^{-2} S \right| \leq W, \quad (4.11)$$

for all residue classes y to moduli $q \leq Q^2$.

In order to define $\hat{\alpha}(x)$ we will need an analogue of the coprimality condition $\gcd(a, q) = 1$ which occurs in (4.6). It turns out that the correct generalisation is to require that there is no non-trivial common divisor of y and q in \mathbb{N} . Of course this is not the same as requiring $\gcd(y, q) = 1$ in \mathfrak{o}_L . We write

$$\sum_{y \pmod{q}}^*$$

to denote a sum in which $y \in \mathfrak{o}_L$ runs over residue classes modulo q such that y and q have no non-trivial common divisor in \mathbb{N} .

We also need an analogue for the exponential function $e_q(a)$. Recall that $\{1, \tau\}$ is a \mathbb{Z} -basis for \mathfrak{o}_L , and hence also a \mathbb{Q} -basis for L . If $x = a + b\tau \in L$, with $a, b \in \mathbb{Q}$, and if $q \in \mathbb{N}$, we define

$$e^{(L)}(x) := e(b) \quad \text{and} \quad e_q^{(L)}(x) := e_q(b).$$

These exponential functions have the property that if $y \in \mathfrak{o}_L$ then

$$\sum_{x \pmod{q}} e_q^{(L)}(xy) = \begin{cases} q^2, & \text{if } q \mid y, \\ 0, & \text{if } q \nmid y. \end{cases}$$

Thus, for the analogue of the Ramanujan sum we have

$$\sum_{x \pmod{q}}^* e_q^{(L)}(xy) = \sum_{d \mid \gcd(q, y)} d^2 \mu(q/d). \quad (4.12)$$

We are now ready to specify our approximation to $\alpha(x)$. We set

$$\hat{\alpha}(x) := \omega(x) \sum_{q \leq Q} \sum_{y \pmod{q}}^* e_q^{(L)}(-yx) \sum_{z \pmod{q}} \rho(z, q) e_q^{(L)}(yz). \quad (4.13)$$

An argument precisely analogous to that used for Lemma 5 then produces the following result.

Lemma 7. — *With the above assumptions we have*

$$|S(z, h) - \hat{S}(z, h)| \leq WQ^4 + E$$

for all $h \leq Q$ and all residue classes z modulo h .

To produce an analogue of Lemma 6 we shall assume that

$$R = \{a + b\tau \in \mathfrak{o}_L : |a - a_0|, |b - b_0| < N\}$$

for certain a_0, b_0 . Then we will have a large sieve inequality for \mathfrak{o}_L , taking the form

$$\sum_{q \leq Q} \sum_{y \pmod{q}}^* \left| \sum_{x \in R} c_x e_q^{(L)}(xy) \right|^2 \leq (\sqrt{2N} + Q)^4 \sum_{x \in R} |c_x|^2. \quad (4.14)$$

This follows from the two dimensional large sieve of Huxley [20, Theorem 1]. Under the condition that $Q^2 \leq N$, which we now impose, we will then have $(\sqrt{2N} + Q)^4 \ll N^2$. We proceed to argue as before to deduce from the dual of the above estimate that

$$\sum_{x \in R} |\hat{\alpha}(x)|^2 \ll \omega_0^2 N^2 \sum_{q \leq Q} \sum_{y \pmod{q}}^* |c_{y,q}|^2,$$

with

$$c_{y,q} = \sum_{z \pmod{q}} \rho(z, q) e_q^{(L)}(yz).$$

This time the estimate (4.8) yields

$$|Sc_{y,q} - T(y, q)| \leq q^2 E,$$

with

$$T(y, q) = \sum_{x \in R} \alpha(x) e_q^{(L)}(xy).$$

Continuing as before we then obtain

$$|S|^2 \sum_{x \in R} |\hat{\alpha}(x)|^2 \ll \omega_0^2 N^2 \left(E^2 Q^7 + N^2 \sum_{x \in R} |\alpha(x)|^2 \right).$$

This gives us the following conclusion.

Lemma 8. — *Suppose that*

$$R = \{a + b\tau \in \mathfrak{o}_L : |a - a_0|, |b - b_0| < N\}$$

for certain a_0, b_0 , and assume that there is a constant ω_0 such that $|\omega(x)| \leq \omega_0$ for all $x \in R$. Then

$$\sum_{x \in R} |\hat{\alpha}(x)|^2 \ll \left(\frac{\omega_0 N^2}{|S|} \right)^2 \left(N^2 + \sum_{x \in R} |\alpha(x)|^2 \right),$$

providing that $Q^2 \leq N$ and $E^2 Q^7 \leq N^4$.

5. The functions $\hat{\alpha}(x)$ and $\alpha_0(x)$

In this section we will apply Lemmas 7 and 8 to the function (3.15). It will be important for us to produce functions $\hat{\alpha}$ and ω which depend only on the set \mathcal{U} , and not on the set R , since we require results which are uniform in R .

We will write $x \in \mathfrak{o}_L$ in the form $x = x_1 + x_2\tau$ with $x_1, x_2 \in \mathbb{Z}$. We shall also write $\delta \mathbf{N}_{K/L}(\mathbf{u})$ as $\mathbf{N}_1(\mathbf{u}) + \mathbf{N}_2(\mathbf{u})\tau$. The reader should observe that this notation does not coincide with that used temporarily in (2.1), nor that used in our discussion of (3.12). We let R be a square in the (x_1, x_2) -plane, with sides parallel to the x_1 and x_2 axes, and side-length $\rho \ll U^{n/2}$. This corresponds, by abuse of notation, to the square R used in defining the sums $S(y, q)$ in (4.7). Extending this abuse of notation we shall allow $\delta \mathbf{N}_{K/L}(\mathbf{u})$ to denote the ordered pair $(\mathbf{N}_1(\mathbf{u}), \mathbf{N}_2(\mathbf{u}))$.

We will specify the function $\omega(x) = \omega(x_1, x_2)$ and verify its properties at the end of this section. For the time being we content ourselves with describing the key features as follows.

Lemma 9. — *There is a continuously differentiable function $\omega : \mathbb{R}^2 \rightarrow [0, \infty)$, depending on \mathcal{U} , for which $\omega(\mathbf{x} + \mathbf{h}) - \omega(\mathbf{x}) \ll U^{-n/2}|\mathbf{h}|$ for all $\mathbf{x}, \mathbf{h} \in \mathbb{R}^2$, and such that*

$$\int_R \omega(x_1, x_2) dx_1 dx_2 = M^n \text{meas}\{\mathbf{u} \in \mathcal{U} : \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\}, \quad (5.1)$$

for every square R as above. Furthermore ω is supported on a disc of radius $O(U^{n/2})$ and satisfies $\omega(\mathbf{x}) \ll 1$ throughout this disc. Moreover there is a disc of radius $\gg G^{-1}U^{n/2}$ around the point $U^{n/2}\delta\mathbf{N}_{K/L}(\mathbf{u}^{(\mathbb{R})})$ on which we have $\omega(\mathbf{x}) \gg G^{2-n}$.

Although it is a real measure which occurs on the right-hand-side of (5.1) what occurs naturally for us is the corresponding cardinality

$$S_0 := \#\{\mathbf{u} \in \mathcal{U} \cap \mathbb{Z}^n : \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\}.$$

We proceed to establish a relation between the two. For each $\mathbf{u} \in \mathcal{U} \cap \mathbb{Z}^n$ we let

$$S(\mathbf{u}) := \{\mathbf{y} \in \mathbb{R}^n : u_i \leq y_i < u_i + 1, (1 \leq i \leq n)\}$$

and

$$\mathcal{U}^{(-)} := \bigcup \{S(\mathbf{u}) : \mathbf{u} \in \mathbb{Z}^n, S(\mathbf{u}) \subseteq \mathcal{U}\}.$$

Thus $\mathcal{U}^{(-)} \subseteq \mathcal{U}$, and the number of integer vectors in \mathcal{U} but not $\mathcal{U}^{(-)}$ is $O(U^{n-1})$. In particular,

$$S_0 = \#\{\mathbf{u} \in \mathcal{U}^{(-)} \cap \mathbb{Z}^n : \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\} + O(U^{n-1}). \quad (5.2)$$

Now if $\mathbf{u} \in \mathcal{U}^{(-)} \cap \mathbb{Z}^n$ and $\mathbf{y} \in S(\mathbf{u})$ then $\delta\mathbf{N}_{K/L}(\mathbf{y}) = \delta\mathbf{N}_{K/L}(\mathbf{u}) + O(U^{n/2-1})$. Thus there are squares R_1 and R_2 of sides ρ_1 and ρ_2 respectively, such that

$$|\rho_1 - \rho| \ll U^{n/2-1} \quad \text{and} \quad |\rho_2 - \rho| \ll U^{n/2-1},$$

and for which

$$\delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_1 \Rightarrow \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R \Rightarrow \delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_2$$

whenever $\mathbf{y} \in S(\mathbf{u})$. We deduce that

$$\begin{aligned} \#\{\mathbf{u} \in \mathcal{U}^{(-)} \cap \mathbb{Z}^n : \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\} \\ \geq \text{meas}\{\mathbf{y} \in \mathcal{U}^{(-)} : \delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_1\} \end{aligned}$$

and

$$\begin{aligned} \#\{\mathbf{u} \in \mathcal{U}^{(-)} \cap \mathbb{Z}^n : \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\} \\ \leq \text{meas}\{\mathbf{y} \in \mathcal{U}^{(-)} : \delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_2\}. \end{aligned}$$

However $\mathcal{U}^{(-)}$ is contained in \mathcal{U} and differs from it by a set of measure $O(U^{n-1})$, whence

$$\begin{aligned} \text{meas}\{\mathbf{y} \in \mathcal{U}^{(-)} : \delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_1\} \\ \geq \text{meas}\{\mathbf{y} \in \mathcal{U} : \delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_1\} + O(U^{n-1}) \end{aligned}$$

and

$$\text{meas}\{\mathbf{y} \in \mathcal{U}^{(-)} : \delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_2\} \leq \text{meas}\{\mathbf{y} \in \mathcal{U} : \delta\mathbf{N}_{K/L}(\mathbf{y}) \in R_2\}.$$

According to (5.1) we may then deduce that

$$\#\{\mathbf{u} \in \mathcal{U}^{(-)} \cap \mathbb{Z}^n : \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\} \geq M^{-n} \int_{R_1} \omega(x_1, x_2) dx_1 dx_2 + O(U^{n-1})$$

and

$$\#\{\mathbf{u} \in \mathcal{U}^{(-)} \cap \mathbb{Z}^n : \delta \mathbf{N}_{K/L}(\mathbf{u}) \in R\} \leq M^{-n} \int_{R_2} \omega(x_1, x_2) dx_1 dx_2.$$

However the squares R_1 and R_2 each differ from R by a set of measure $O(U^{n-1})$, and furthermore $\omega \ll 1$. Thus

$$\#\{\mathbf{u} \in \mathcal{U}^{(-)} \cap \mathbb{Z}^n : \delta \mathbf{N}_{K/L}(\mathbf{u}) \in R\} = M^{-n} \int_R \omega(x_1, x_2) dx_1 dx_2 + O(U^{n-1}).$$

Finally, if we compare this with (5.2), we deduce that

$$S_0 = M^{-n} \int_R \omega(x_1, x_2) dx_1 dx_2 + O(U^{n-1}). \quad (5.3)$$

We next consider

$$\sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{y} \pmod{q}}} \omega(\mathbf{x}).$$

To each point \mathbf{x} counted in the above sum we associate the square $R(\mathbf{x}) = [x_1, x_1 + q) \times [x_2, x_2 + q)$. If $\mathbf{t} \in R(\mathbf{x})$ then $\omega(\mathbf{t}) - \omega(\mathbf{x}) \ll qU^{-n/2}$ by Lemma 9, whence

$$\int_{R(\mathbf{x})} \omega(\mathbf{t}) dt_1 dt_2 = q^2 \omega(\mathbf{x}) + O(q^3 U^{-n/2}).$$

We sum this for points $\mathbf{x} \in R$ with $\mathbf{x} \equiv \mathbf{y} \pmod{q}$. Since there are $O(q^{-2}U^n)$ such points if $q \leq U^{n/2}$ we deduce that

$$\sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{y} \pmod{q}}} \omega(\mathbf{x}) = q^{-2} \sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{y} \pmod{q}}} \int_{R(\mathbf{x})} \omega(\mathbf{t}) dt_1 dt_2 + O(qU^{n/2}).$$

The union of the squares $R(\mathbf{x})$ will be a square R' say, whose sides are within a distance q of the sides of R . Thus R and R' differ by a set of measure $O(qU^{n/2})$, if $q \leq U^{n/2}$. Since $\omega(\mathbf{t}) \ll 1$ for all \mathbf{t} this produces

$$\begin{aligned} \sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{y} \pmod{q}}} \omega(\mathbf{x}) &= q^{-2} \int_{R'} \omega(\mathbf{t}) dt_1 dt_2 + O(qU^{n/2}) \\ &= q^{-2} \int_R \omega(\mathbf{t}) dt_1 dt_2 + O(qU^{n/2}). \end{aligned}$$

Comparing this with (5.3) we therefore obtain

$$M^{-n} \sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{y} \pmod{q}}} \omega(\mathbf{x}) = q^{-2} S_0 + O(U^{n-1}),$$

providing that $n \geq 4$ and $q \leq U$. In particular, when $q = 1$ we obtain

$$S = M^n S_0 + O(U^{n-1}), \quad (5.4)$$

so that (4.11) holds with $W \ll U^{n-1}$ and any $Q \leq U^{1/2}$.

We next consider the condition (4.8). In view of the definition (3.15) we see that

$$S(y, q) = \sum_{\mathbf{w}} \#\{\mathbf{u} \in \mathcal{U} \cap \mathbb{Z}^n : \mathbf{u} \equiv \mathbf{w} \pmod{[M, q]}, \delta \mathbf{N}_{K/L}(\mathbf{u}) \in R\},$$

where \mathbf{w} runs over vectors modulo $[M, q]$ for which $\mathbf{w} \equiv \mathbf{u}^{(M)} \pmod{M}$ and $\delta\mathbf{N}_{K/L}(\mathbf{w}) \equiv y \pmod{q}$. For a general modulus $r \leq U$ we proceed to compare the sizes of the sets

$$T(r, \mathbf{x}) := \{\mathbf{u} \in \mathcal{U} \cap \mathbb{Z}^n : \mathbf{u} \equiv \mathbf{x} \pmod{r}, \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\}$$

for the values $\mathbf{x} = \mathbf{w}$ and $\mathbf{0}$. For $\mathbf{u} = (u_1, \dots, u_n)$ let

$$\mathbf{u}^* = (r[u_1/r], \dots, r[u_n/r]).$$

If $\mathbf{u} \in T(r, \mathbf{w})$ then \mathbf{u}^* will belong to $T(r, \mathbf{0})$ unless either \mathbf{u} is within a distance $O(r)$ of the boundary of \mathcal{U} , or $\delta\mathbf{N}_{K/L}(\mathbf{u})$ is within a distance $O(rU^{n/2-1})$ of the boundary of R . A pair (N_1, N_2) arises at most $O_\eta(U^\eta)$ times as a value of $\delta\mathbf{N}_{K/\mathbb{Q}}(\mathbf{u})$ with $\mathbf{u} \in \mathcal{U} \cap \mathbb{Z}^n$, and hence it follows that

$$\#T(r, \mathbf{0}) = \#T(r, \mathbf{w}) + O_\eta(rU^{n-1+\eta}) \quad (5.5)$$

for any $\eta > 0$. If we now sum for all $\mathbf{w} \pmod{r}$ we deduce that

$$r^n \#T(r, \mathbf{0}) = \#\{\mathbf{u} \in \mathcal{U} \cap \mathbb{Z}^n : \delta\mathbf{N}_{K/L}(\mathbf{u}) \in R\} + O_\eta(r^{n+1}U^{n-1+\eta})$$

and hence that

$$r^n \#T(r, \mathbf{0}) = S_0 + O_\eta(r^{n+1}U^{n-1+\eta}).$$

Substituting this back into (5.5) we deduce that

$$\#T(r, \mathbf{w}) = r^{-n}S_0 + O_\eta(rU^{n-1+\eta}).$$

We therefore see that

$$S(y, q) = M^{-n}\rho(y, q)S_0 + O_\eta(q^{n+1}U^{n-1+\eta}),$$

with

$$\rho(y, q) := \frac{M^n}{[M, q]^n} \# \left\{ \mathbf{s} \pmod{[M, q]} : \begin{array}{l} \mathbf{s} \equiv \mathbf{u}^{(M)} \pmod{M}, \\ \delta\mathbf{N}_{K/L}(\mathbf{s}) \equiv y \pmod{q} \end{array} \right\}. \quad (5.6)$$

The conditions (4.9) and (4.10) are now readily checked, and we see that (4.8) follows from (5.4) with $E \ll_\eta Q^{n+1}U^{n-1+\eta}$.

We are now ready to apply Lemma 7, which produces the following result.

Lemma 10. — *Let R be a square with side $\rho \geq 1$ satisfying $\rho \ll U^{n/2}$. Then if η is any positive constant we have*

$$\sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \ll_\eta Q^{n+1}U^{n-1+\eta},$$

for all $q \leq Q \leq U^{1/2}$ and all $y \pmod{q}$.

In our application the square R will vary and so it is crucial that Lemma 10 is uniform in squares of side length $O(U^{n/2})$. As to Lemma 8, we will only need a result for the L^2 -norm taken over all x . Hence we choose R to be a square centred at the origin and with side $\rho = cU^{n/2}$, in which the constant c is taken sufficiently large that $x \in R$ whenever $\hat{\alpha}(x) \neq 0$. We may clearly take $\omega_0 \ll 1$ by Lemma 9 and

$$\sum_{x \in R} |\alpha(x)|^2 \ll_\eta U^{n+\eta}$$

by Lemma 4. Our remaining task is thus to estimate S from below. However our choice of R ensures that $S_0 = \#(\mathcal{U} \cap \mathbb{Z}^n) \gg U^n G^{-n}$, whence (5.4) yields $S \gg U^n G^{-n}$, assuming that $G \leq U^{1/(n+1)}$, say. Consequently we deduce from Lemma 8 the following bound.

Lemma 11. — *For any constant $\eta > 0$ we have*

$$\sum_{x \in \mathfrak{o}_L} |\hat{\alpha}(x)|^2 \ll_{\eta} U^{n+\eta} G^{O(1)}$$

providing that $Q^{n+5} \leq U^{1-\eta}$ and $G \leq U^{1/(n+1)}$. Moreover we then have

$$\sum_{x \in \mathfrak{o}_L} |\alpha_0(x)|^2 \ll_{\eta} U^{n+\eta} G^{O(1)}.$$

The final estimate follows immediately from Lemma 4, since

$$\sum_{x \in \mathfrak{o}_L} |\alpha_0(x)|^2 \ll \sum_{x \in \mathfrak{o}_L} |\alpha(x)|^2 + \sum_{x \in \mathfrak{o}_L} |\hat{\alpha}(x)|^2.$$

We end this section by proving Lemma 9. We first show that the map from \mathbb{C}^n to \mathbb{C}^2 given by $\mathbf{u} \mapsto (\mathbf{N}_1(\mathbf{u}), \mathbf{N}_2(\mathbf{u}))$ is non-singular at any point for which $\mathbf{N}_{K/\mathbb{Q}}(\mathbf{u}) \neq 0$, by which we mean that $\nabla \mathbf{N}_1(\mathbf{u})$ and $\nabla \mathbf{N}_2(\mathbf{u})$ are only proportional if $\mathbf{N}_{K/\mathbb{Q}}(\mathbf{u}) = 0$. Clearly it suffices to do the same for the map

$$\mathbf{u} \mapsto (\mathbf{N}_1(\mathbf{u}) + \tau \mathbf{N}_2(\mathbf{u}), \mathbf{N}_1(\mathbf{u}) + \tau^{\sigma} \mathbf{N}_2(\mathbf{u})).$$

However $\mathbf{N}_1(\mathbf{u}) + \tau \mathbf{N}_2(\mathbf{u})$ (respectively, $\mathbf{N}_1(\mathbf{u}) + \tau^{\sigma} \mathbf{N}_2(\mathbf{u})$) is a product of δ (respectively, δ^{σ}) with certain conjugates of $u_1 \omega_1 + \cdots + u_n \omega_n$. Thus we can write our map in the form

$$\mathbf{u} \mapsto (\Lambda_1(\mathbf{u}) \cdots \Lambda_{n/2}(\mathbf{u}), \Lambda_{n/2+1}(\mathbf{u}) \cdots \Lambda_n(\mathbf{u})),$$

with suitable linear forms Λ_i , which will be linearly independent. Hence if we set $v_i = \Lambda_i(\mathbf{u})$ for $1 \leq i \leq n$ it will suffice to show that the map

$$\mathbf{v} \mapsto (v_1 \cdots v_{n/2}, v_{n/2+1} \cdots v_n)$$

is non-singular whenever $v_1 \cdots v_n \neq 0$. This however is trivial.

Since $\mathbf{N}_{K/\mathbb{Q}}(\mathbf{u}^{(\mathbb{R})}) \neq 0$, we deduce that there are indices i and j such that

$$\frac{\partial \mathbf{N}_1(\mathbf{u}^{(\mathbb{R})})}{\partial u_i} \frac{\partial \mathbf{N}_2(\mathbf{u}^{(\mathbb{R})})}{\partial u_j} > \frac{\partial \mathbf{N}_1(\mathbf{u}^{(\mathbb{R})})}{\partial u_j} \frac{\partial \mathbf{N}_2(\mathbf{u}^{(\mathbb{R})})}{\partial u_i}.$$

We suppose for notational simplicity that $i = 1$ and $j = 2$, the other cases being similar. By continuity we then have

$$J(\mathbf{u}) := \frac{\partial \mathbf{N}_1(\mathbf{u})}{\partial u_1} \frac{\partial \mathbf{N}_2(\mathbf{u})}{\partial u_2} - \frac{\partial \mathbf{N}_1(\mathbf{u})}{\partial u_2} \frac{\partial \mathbf{N}_2(\mathbf{u})}{\partial u_1} \gg U^{n-2} \quad (5.7)$$

throughout \mathcal{U} , if G is large enough. We now write $\mathbf{v} = (u_1, u_2)$ and $\mathbf{u} = (\mathbf{v}, \mathbf{w})$, where $\mathbf{w} = (u_3, \dots, u_n)$. We will find it convenient to number the entries in \mathbf{w} as (w_3, \dots, w_n) . By the Implicit Function Theorem, if

$$\mathbf{N}_1(\mathbf{v}, \mathbf{w}) = x_1, \quad \mathbf{N}_2(\mathbf{v}, \mathbf{w}) = x_2$$

with $(\mathbf{v}, \mathbf{w}) \in \mathcal{U}$, we can express \mathbf{v} in terms of \mathbf{x} and \mathbf{w} as $\mathbf{v} = \mathbf{v}(\mathbf{x}, \mathbf{w})$.

We may now calculate that

$$\begin{aligned} & \text{meas}\{\mathbf{u} \in \mathcal{U} : \delta \mathbf{N}_{K/L}(\mathbf{u}) \in R\} \\ &= \int_{\mathbf{x} \in R} \int_{\mathbf{w} \in W(\mathbf{x})} J(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})^{-1} dw_3 \cdots dw_n dx_1 dx_2, \end{aligned}$$

where

$$W(\mathbf{x}) := \{\mathbf{w} \in \mathbb{R}^{n-2} : (\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w}) \in \mathcal{U}\}.$$

We therefore define

$$\omega(\mathbf{x}) := M^n \int_{\mathbf{w} \in W(\mathbf{x})} J(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})^{-1} dw_3 \cdots dw_n, \quad (5.8)$$

so that (5.1) immediately follows. It is clear from the definition of $W(\mathbf{x})$ that ω is supported on the set of values $\delta \mathbf{N}_{K/L}(\mathbf{u})$ as \mathbf{u} runs over \mathcal{U} . Thus the support is contained in a disc of radius $O(U^{n/2})$ as claimed in the lemma. Moreover

$$\text{meas}\{W(\mathbf{x})\} \leq \text{meas}\{\mathbf{w} \in \mathbb{R}^{n-2} : (\mathbf{v}, \mathbf{w}) \in \mathcal{U} \text{ for some } \mathbf{v}\} \ll U^{n-2}, \quad (5.9)$$

whence (5.7) yields $\omega(\mathbf{x}) \ll 1$ as required.

The function $\mathbf{v}(\mathbf{x}, \mathbf{w})$ will be continuously differentiable with respect to \mathbf{x} and \mathbf{w} . Moreover it will be weighted-homogeneous of degree 1 in \mathbf{w} and degree $2/n$ in \mathbf{x} . Since $\partial x_k / \partial v_l \ll U^{n/2-1}$ for $1 \leq k, l \leq 2$ we deduce from (5.7) that

$$\frac{\partial v_k}{\partial x_l} \ll J(\mathbf{u})^{-1} U^{n/2-1} \ll U^{1-n/2}, \quad (1 \leq k, l \leq 2). \quad (5.10)$$

We observe from (3.9) that $\partial \mathbf{N}_{K/\mathbb{Q}}(\mathbf{u}^{(\mathbb{R})}) / \partial w_3$ does not vanish, whence

$$\partial \mathbf{N}_j(\mathbf{u}^{(\mathbb{R})}) / \partial w_3 \neq 0$$

for at least one of $j = 1$ or $j = 2$. Moreover

$$\frac{\partial \mathbf{N}_j(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})}{\partial w_3} = \frac{\partial x_j}{\partial w_3} = 0$$

for $j = 1, 2$, whence

$$\frac{\partial \mathbf{N}_j}{\partial v_1} \frac{\partial v_1}{\partial w_3} + \frac{\partial \mathbf{N}_j}{\partial v_2} \frac{\partial v_2}{\partial w_3} + \frac{\partial \mathbf{N}_j}{\partial w_3} = 0, \quad (j = 1, 2).$$

Thus at least one of $\partial v_1 / \partial w_3$ and $\partial v_2 / \partial w_3$ is non-vanishing at $\mathbf{u}^{(\mathbb{R})}$. We suppose that $\partial v_1 / \partial w_3 \neq 0$, the alternative case being similar. By continuity we then have $|\partial v_1 / \partial w_3| \gg 1$ for $|\mathbf{u} - \mathbf{u}^{(\mathbb{R})}| \ll G^{-1}$, if G is large enough. It follows that

$$\left| \frac{\partial v_1}{\partial w_3} \right| \gg 1 \quad (5.11)$$

throughout \mathcal{U} , since the partial derivative is homogeneous in \mathbf{w} , of weight zero.

We can now investigate $\omega(\mathbf{x} + \mathbf{h}) - \omega(\mathbf{x})$. On $W(\mathbf{x} + \mathbf{h}) \cap W(\mathbf{x})$ we have

$$\begin{aligned} J(\mathbf{v}(\mathbf{x} + \mathbf{h}, \mathbf{w}), \mathbf{w})^{-1} - J(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})^{-1} \\ \ll U^{4-2n} |J(\mathbf{v}(\mathbf{x} + \mathbf{h}, \mathbf{w}), \mathbf{w}) - J(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})|, \end{aligned}$$

by (5.7). Moreover

$$J(\mathbf{v}(\mathbf{x} + \mathbf{h}, \mathbf{w}), \mathbf{w}) - J(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w}) \ll U^{n-3} |\mathbf{v}(\mathbf{x} + \mathbf{h}, \mathbf{w}) - \mathbf{v}(\mathbf{x}, \mathbf{w})|,$$

since $J(\mathbf{u})$ is a form in \mathbf{u} of degree $n - 2$. It then follows from (5.10) that

$$J(\mathbf{v}(\mathbf{x} + \mathbf{h}, \mathbf{w}), \mathbf{w})^{-1} - J(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})^{-1} \ll U^{2-3n/2} |\mathbf{h}|$$

on $W(\mathbf{x} + \mathbf{h}) \cap W(\mathbf{x})$. We therefore see from (5.9) that the contribution to $\omega(\mathbf{x} + \mathbf{h}) - \omega(\mathbf{x})$ from the set $W(\mathbf{x} + \mathbf{h}) \cap W(\mathbf{x})$ will be $O(U^{-n/2} |\mathbf{h}|)$. This is satisfactory.

On the remaining range we merely use the bound $J^{-1} \ll U^{2-n}$. It therefore suffices to estimate the measure of the set of $\mathbf{w} \in \mathbb{R}^{n-2}$ for which either $(\mathbf{v}(\mathbf{x} + \mathbf{h}, \mathbf{w}), \mathbf{w}) \in \mathcal{U}$ or $(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w}) \in \mathcal{U}$, but not both. By substituting $\mathbf{x}' = \mathbf{x} + \mathbf{h}$ and $\mathbf{h}' = -\mathbf{h}$ if necessary, we may suppose that $(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w}) \in \mathcal{U}$ and $(\mathbf{v}(\mathbf{x} + \mathbf{h}, \mathbf{w}), \mathbf{w}) \notin \mathcal{U}$. In view of (5.10) this means that $(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})$ lies in \mathcal{U} at a distance $O(U^{1-n/2}|\mathbf{h}|)$ from the boundary of \mathcal{U} . Let $\mathcal{U}_{\mathbf{x}}(\mathbf{h})$ denote the set of such points \mathbf{w} . As yet we have not completely specified the set \mathcal{U} in (3.14), and it is now time to do so. For indices $i \geq 3$ we merely choose $L_i(\mathbf{u}) = w_i$. Then if $(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})$ lies in \mathcal{U} at a distance $O(U^{1-n/2}|\mathbf{h}|)$ from the edge defined by L_i we see that the corresponding w_i runs over an interval of length $O(U^{1-n/2}|\mathbf{h}|)$, so that the contribution to $\text{meas}(\mathcal{U}_{\mathbf{x}}(\mathbf{h}))$ is $\ll U^{1-n/2}|\mathbf{h}|U^{n-3} = U^{n/2-2}|\mathbf{h}|$. We take the remaining linear forms L_i to be v_1 and $v_1 + \lambda v_2$. Here λ is a non-zero constant chosen sufficiently small that

$$\left| \frac{\partial(v_1 + \lambda v_2)}{\partial w_3} \right| \gg 1$$

throughout \mathcal{U} . In view of (5.11) such a choice will be possible, since we have $\partial v_2 / \partial w_3 \ll 1$. Suppose now that w_4, w_5, \dots, w_n are fixed. Then if $(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w})$ lies in \mathcal{U} at a distance $O(U^{1-n/2}|\mathbf{h}|)$ from the edge defined by the linear form $L_i = v_1$ we see that w_3 is confined to an interval of length $O(U^{1-n/2}|\mathbf{h}|)$, and similarly for the edge defined by $L_i = v_1 + \lambda v_2$. It follows that the contribution to $\text{meas}(\mathcal{U}_{\mathbf{x}}(\mathbf{h}))$ is $O(U^{n/2-2}|\mathbf{h}|)$ in these cases too. Since $J^{-1} \ll U^{2-n}$ we deduce that $\omega(\mathbf{x} + \mathbf{h}) - \omega(\mathbf{x}) \ll U^{-n/2}|\mathbf{h}|$ as required.

It remains to establish the lower bound for $\omega(\mathbf{x})$. It is clear that $J(\mathbf{u})$ is homogeneous of degree $n-2$, whence $J(\mathbf{u}) \ll U^{n-2}$ for all relevant \mathbf{u} . In view of (5.8) it therefore suffices to show that $\text{meas}\{W(\mathbf{x})\} \gg G^{2-n}U^{n-2}$ on a suitable set of values \mathbf{x} . Now

$$\left(\mathbf{v}(\delta \mathbf{N}_{K/L}(\mathbf{u}^{(\mathbb{R})}), u_3^{(\mathbb{R})}, \dots, u_n^{(\mathbb{R})}), u_3^{(\mathbb{R})}, \dots, u_n^{(\mathbb{R})} \right) = \mathbf{u}^{(\mathbb{R})}.$$

Moreover if $|\mathbf{x}| \ll U^{n/2}$ and $|\mathbf{w}| \ll U$ then

$$\begin{aligned} & \left| \mathbf{v}(\mathbf{x}, \mathbf{w}) - \mathbf{v} \left(U^{n/2} \delta \mathbf{N}_{K/L}(\mathbf{u}^{(\mathbb{R})}), U u_3^{(\mathbb{R})}, \dots, U u_n^{(\mathbb{R})} \right) \right| \\ & \ll U \left\{ |U^{-n/2} \mathbf{x} - \delta \mathbf{N}_{K/L}(\mathbf{u}^{(\mathbb{R})})| + \max_{3 \leq i \leq n} |U^{-1} w_i - u_i^{(\mathbb{R})}| \right\}, \end{aligned}$$

by the homogeneity properties of $\mathbf{v}(\mathbf{x}, \mathbf{w})$. We therefore see from the definition (3.14) of \mathcal{U} that there is a small constant $c > 0$ such that $(\mathbf{v}(\mathbf{x}, \mathbf{w}), \mathbf{w}) \in \mathcal{U}$ whenever we have $|w_i - U u_i^{(\mathbb{R})}| \leq c G^{-1} U$ for $3 \leq i \leq n$ and

$$|\mathbf{x} - U^{n/2} \delta \mathbf{N}_{K/L}(\mathbf{u}^{(\mathbb{R})})| \leq c G^{-1} U^{n/2}.$$

We therefore have $\text{meas}\{W(\mathbf{x})\} \gg G^{2-n}U^{n-2}$ in the above region, as required.

6. A large sieve bound for $\alpha_0(x)$

From Lemma 10 we know that $\alpha_0(x)$ is evenly distributed in all congruence classes for moduli $q \leq Q$. The condition that $Q \leq U^{1/2}$ causes no problems. However a more serious constraint on the size of Q comes from the fact that we cannot handle \hat{a} if Q is too large.

The goal of the present section is to show that in fact α_0 is evenly distributed for “almost all” congruence classes, for much larger values of q . This will enable us to get equidistribution

for large moduli, on average, while keeping Q sufficiently small that $\hat{\alpha}$ can be adequately handled.

Our equidistribution result will be achieved by a quite general large sieve argument, motivated by (but, we believe, simpler than) that used by Fouvry and Iwaniec [12, § 10]. A related procedure is given by Iwaniec and Kowalski [21, Theorem 17.5]. However we have found it more convenient to use additive characters directly, rather than to switch to multiplicative ones as they do. The reader might care to note that a somewhat different argument, also of a very general kind, appears in Heath-Brown [17, § 2].

We assume that we have a function $\alpha_0(x)$ defined on \mathfrak{o}_L , satisfying

$$\left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \right| \leq W_0, \quad (6.1)$$

for all $q \leq Q$ and $y \in \mathfrak{o}_L$, where R is a square of side $2N$ as in Lemma 8. For the duration of this section let us write

$$\Sigma(S; q, y) := \sum_{\substack{x \in S \\ x \equiv y \pmod{q}}} \alpha_0(x),$$

for any square S . We proceed to consider

$$S_1(Q_0) := \sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} |\Sigma(R; q, y)|^2.$$

If

$$K(t) := \sum_{x \in R} \alpha_0(x) e^{(L)}(tx)$$

for $t \in L$ then

$$\sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) = q^{-2} \sum_{b \pmod{q}} e_q^{(L)}(-by) K(b/q),$$

and we deduce that

$$S_1(Q_0) = \sum_{q \leq Q_0} \sum_{b \pmod{q}} |K(b/q)|^2.$$

We now write the fraction b/q in lowest terms as c/h , say. Then

$$\begin{aligned} S_1(Q_0) &= \sum_{h \leq Q_0} \sum_{c \pmod{h}}^* |K(c/h)|^2 \# \{q \leq Q_0 : h \mid q\} \\ &\leq Q_0 \sum_{h \leq Q_0} h^{-1} \sum_{c \pmod{h}}^* |K(c/h)|^2. \end{aligned}$$

The contribution from terms $h > Q$ is at most

$$Q_0 Q^{-1} \sum_{h \leq Q_0} \sum_{c \pmod{h}}^* |K(c/h)|^2 \leq Q_0 Q^{-1} (\sqrt{2N} + Q_0)^4 \sum_{x \in R} |\alpha_0(x)|^2,$$

by the two dimensional large sieve in the form (4.14).

For the remaining terms with $h \leq Q$ we observe that

$$\begin{aligned} \sum_{c \pmod{h}}^* |K(c/h)|^2 &\leq \sum_{c \pmod{h}} |K(c/h)|^2 = h^2 \sum_{y \pmod{h}} |\Sigma(R; q, y)|^2 \\ &\leq h^4 W_0^2, \end{aligned}$$

by our assumption (6.1). Thus terms with $h \leq Q$ contribute at most $Q_0 Q^4 W_0^2$ to $S_1(Q_0)$. This enables us to conclude as follows.

Lemma 12. — *Under the assumption (6.1) we have*

$$\sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} \left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \right|^2 \ll Q_0 Q^4 W_0^2 + \frac{Q_0(N^2 + Q_0^4)}{Q} \sum_{x \in R} |\alpha_0(x)|^2.$$

We will require a form of this estimate in which we have a maximum over different squares R . We assume that $\alpha_0(x)$ is supported on a set

$$S = \{a + b\tau \in \mathfrak{o}_L : (a, b) \in (-N, N]^2\}$$

and proceed to cover this with K^2 smaller squares, each contained in S and of the type

$$R_i = \{a + b\tau \in \mathfrak{o}_L : (a - a_i, b - b_i) \in (-N/K, N/K]^2\},$$

for appropriate pairs (a_i, b_i) . Here $K \leq N$ is a positive integer parameter which we will specify in due course. Now any square $R \subseteq S$, with sides aligned with the coordinate axes, will include a union of certain of the squares R_i , outside which there are only $O(N^2 K^{-1})$ points of \mathfrak{o}_L . If we now require that

$$|\alpha_0(x)| \leq A_0 \tag{6.2}$$

for all x , then

$$|\Sigma(R; q, y)| \leq O(N^2 K^{-1} A_0) + \sum_{i \leq K^2} |\Sigma(R_i; q, y)|.$$

Thus if

$$S_2(Q_0) := \sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} \max_R |\Sigma(R; q, y)|^2,$$

then

$$S_2(Q_0) \ll N^4 K^{-2} A_0^2 Q_0^5 + K^2 \sum_{i \leq K^2} \sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} |\Sigma(R_i; q, y)|^2.$$

Lemma 12 then yields

$$\sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} |\Sigma(R_i; q, y)|^2 \ll Q_0 Q^4 W_0^2 + \frac{Q_0(N^2 K^{-2} + Q_0^4)}{Q} \sum_{x \in R_i} |\alpha_0(x)|^2.$$

Since

$$\sum_{i \leq K^2} \sum_{x \in R_i} |\alpha_0(x)|^2 \leq \sum_{x \in R} |\alpha_0(x)|^2,$$

we deduce that

$$S_2(Q_0) \ll N^4 K^{-2} A_0^2 Q_0^5 + K^4 Q_0 Q^4 W_0^2 + \frac{K^2 Q_0(N^2 K^{-2} + Q_0^4)}{Q} \sum_{x \in S} |\alpha_0(x)|^2.$$

We now choose $K = A_0 Q_0^3$, which yields the following conclusion.

Lemma 13. — *Under the assumptions (6.1) and (6.2) we have*

$$\begin{aligned} \sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} \max_R \left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \right|^2 \\ \ll N^4 Q_0^{-1} + A_0^4 Q_0^{13} Q^4 W_0^2 + Q_0 Q^{-1} N^2 \sum_{x \in S} |\alpha_0(x)|^2, \end{aligned}$$

providing that $A_0 Q_0^5 \leq N$.

We apply this last result to our particular situation. In view of the conditions in Lemmas 10 and 11 we take N of order $U^{n/2}$, $Q^{n+5} \leq U^{1-\eta}$ and $G \leq U^{1/(n+1)}$, so that the value $W_0 = Q^{n+1} U^{n-1+\eta}$ is admissible. In order to estimate $\alpha_0(x)$ we note that $\alpha(x) \ll_\eta U^\eta$ for any $\eta > 0$. Moreover (4.13) yields $|\hat{\alpha}(x)| \leq Q^3 \omega(x)$, since (4.9) and (4.10) imply

$$\sum_{z \pmod{q}} \rho(z, q) = 1. \quad (6.3)$$

Since $\omega(x) \ll 1$ by Lemma 9, we will certainly have $\alpha_0(x) \ll_\eta Q^3 U^\eta$. Taking $A_0 \ll_\eta Q^3 U^\eta$, the right hand side in Lemma 13 is therefore

$$\ll_\eta U^{2n} Q_0^{-1} + Q_0^{13} Q^{2n+18} U^{2n-2+O(\eta)} + Q_0 Q^{-1} U^{2n+\eta} G^{O(1)},$$

by Lemma 11. This enables us to conclude as follows.

Lemma 14. — *If $Q \leq Q_0 \leq U^{1/(n+16)}$ and $G \leq U^{1/(n+1)}$, then*

$$\sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} \max_R \left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \right|^2 \ll_\eta Q_0 Q^{-1} U^{2n+O(\eta)} G^{O(1)},$$

for any $\eta > 0$.

We end by establishing a trivial bound for the above sum, which provides an instructive comparison. If $q \leq Q_0 \leq U^{n/2}$ then

$$\begin{aligned} \left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \right|^2 &\ll \#\{x \in R : x \equiv y \pmod{q}\} \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} |\alpha_0(x)|^2 \\ &\ll U^n q^{-2} \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} |\alpha_0(x)|^2, \end{aligned}$$

whence

$$\begin{aligned} \sum_{y \pmod{q}} \max_R \left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \right|^2 &\ll U^n q^{-2} \sum_{y \pmod{q}} \sum_{x \equiv y \pmod{q}} |\alpha_0(x)|^2 \\ &= U^n q^{-2} \sum_{x \in \mathfrak{o}_L} |\alpha_0(x)|^2. \end{aligned} \quad (6.4)$$

We therefore obtain the trivial bound

$$\sum_{q \leq Q_0} q^2 \sum_{y \pmod{q}} \max_R \left| \sum_{\substack{x \in R \\ x \equiv y \pmod{q}}} \alpha_0(x) \right|^2 \ll_{\eta} Q_0 U^{2n+\eta} G^{O(1)},$$

via Lemma 11. Thus Lemma 14 provides a saving which is a power of Q , providing that Q is larger than a suitable power of $U^\eta G$.

7. Bilinear forms in dimension 2

The estimation of bilinear forms is one of the cornerstones of analytic number theory and can be traced back to the work of Vinogradov. Given finite sequences $u_m, v_n \in \mathbb{C}$ and a matrix $\mathbf{A} = (a_{m,n})$ of complex numbers, the essential problem is to estimate the double sum

$$\mathbf{u}^T \mathbf{A} \mathbf{v} = \sum_m \sum_n u_m v_n a_{m,n}.$$

The standard procedure is to use Cauchy's inequality to remove the dependence on $\mathbf{u} = (u_m)$. This gives

$$|\mathbf{u}^T \mathbf{A} \mathbf{v}| \leq \left(\sum_m |u_m|^2 \right)^{1/2} \left(\sum_m \left| \sum_n v_n a_{m,n} \right|^2 \right)^{1/2}.$$

For the second term on the right one expands the square and reverses the order of summation, so as to use suitable information about the sum

$$\sum_m a_{m,n_1} \overline{a_{m,n_2}}.$$

It may happen that the sum itself is small. In other cases one can give an asymptotic evaluation with a main term $M(n_1, n_2)$ say. One may then complete the analysis via an estimation of the sum $\sum_{n_1} \sum_{n_2} v_{n_1} \overline{v_{n_2}} M(n_1, n_2)$.

In our work we will require an analogue of this procedure for sequences indexed by elements of \mathbb{Z}^2 , rather than by \mathbb{Z} . In this endeavour we are inspired by the analytic machinery developed for the Gaussian integers $\mathbb{Z}[i]$ by Fouvry and Iwaniec [12, § 9]. We will provide a completely self-contained account of the method, which has the advantage of being slightly more general in scope.

Let $\mathbf{M} \in \mathrm{SL}_2(\mathbb{Z})$ and let $\alpha, \beta : \mathbb{Z}^2 \rightarrow \mathbb{C}$ be arbitrary functions with finite L^2 -norms

$$\|\alpha\|_2 := \left(\sum_{\mathbf{x} \in \mathbb{Z}^2} |\alpha(\mathbf{x})|^2 \right)^{1/2}, \quad \|\beta\|_2 := \left(\sum_{\mathbf{y} \in \mathbb{Z}^2} |\beta(\mathbf{y})|^2 \right)^{1/2}.$$

Furthermore, let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ be such that

$$\|\lambda\|_2 := \left(\sum_{l \in \mathbb{Z}} |\lambda(l)|^2 \right)^{1/2}$$

is also finite. Our objective is to estimate the double sum

$$S(\alpha, \beta; \lambda) := \sum_{\mathbf{x} \in \mathbb{Z}^2} \sum_{\mathbf{y} \in \mathbb{Z}^2}^{(E)} \alpha(\mathbf{x}) \beta(\mathbf{y}) \lambda(\mathbf{x}^T \mathbf{M} \mathbf{y}),$$

where $\sum^{(E)}$ indicates that the summation over \mathbf{y} is restricted by the condition $\gcd(y_1, y_2) \leq E$.

In the analysis of this section the implied constants in our $O(\cdot)$ and \ll notation will be allowed to depend implicitly on the coefficients of the matrix \mathbf{M} . However they will be uniform in all other parameters. It will be convenient to introduce a norm on elements $\mathbf{x} \in \mathbb{R}^2$ via $|\mathbf{x}| = \max\{|x_1|, |x_2|\}$. We should observe that this particular choice of norm is important, since it will be significant for us that the unit ball is a square.

Let $A, B \geq 1$ with $A \geq B$. We will think of A and B as being large, with A/B also large, but of considerably smaller order than B . We will suppose that α, β are supported on the sets

$$\mathcal{A} = \{\mathbf{x} \in \mathbb{Z}^2 : |\mathbf{x}| \leq A\}, \quad \mathcal{B} = \{\mathbf{y} \in \mathbb{Z}^2 : B \leq |\mathbf{y}| \leq 2B\}, \quad (7.1)$$

respectively. Let M_0 denote the maximum of the moduli of the entries of \mathbf{M} , and let

$$\widetilde{\mathbf{M}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{M}.$$

We now define

$$\widetilde{B} = \{\mathbf{z} \in \mathbb{Z}^2 : (2M_0)^{-1}B \leq |\mathbf{z}| \leq 4M_0B\}$$

and we observe that $\widetilde{\mathbf{M}}\mathbf{y} \in \widetilde{B}$ whenever $\mathbf{y} \in \mathcal{B}$.

An application of Cauchy's inequality now gives

$$|S(\alpha, \beta; \lambda)|^2 \leq \|\beta\|_2^2 \cdot \|\lambda\|_2^2 \sum_{\mathbf{y} \in \mathcal{B}} \sum_{l \in \mathbb{Z}}^{(E)} \left| \sum_{\substack{\mathbf{x} \in \mathbb{Z}^2 \\ \mathbf{x}^T \mathbf{M} \mathbf{y} = l}} \alpha(\mathbf{x}) \right|^2.$$

Enlarging the range of summation for \mathbf{y} we therefore deduce that

$$|S(\alpha, \beta; \lambda)| \leq \|\beta\|_2 \cdot \|\lambda\|_2 \cdot T(\alpha)^{1/2}, \quad (7.2)$$

where

$$T(\alpha) := \sum_{\substack{\mathbf{y} \in \mathbb{Z}^2 \\ \widetilde{\mathbf{M}}\mathbf{y} \in \widetilde{B}}}^{(E)} \sum_{l \in \mathbb{Z}} \left| \sum_{\substack{\mathbf{x} \in \mathbb{Z}^2 \\ \mathbf{x}^T \mathbf{M} \mathbf{y} = l}} \alpha(\mathbf{x}) \right|^2.$$

Opening up the inner sum we obtain

$$T(\alpha) = \sum_{\substack{\mathbf{y} \in \mathbb{Z}^2 \\ \widetilde{\mathbf{M}}\mathbf{y} \in \widetilde{B}}}^{(E)} \sum_{\substack{\mathbf{w} \in \mathbb{Z}^2 \\ \mathbf{w}^T \mathbf{M} \mathbf{y} = 0}} (\alpha * \alpha)(\mathbf{w}),$$

with

$$(\alpha * \alpha)(\mathbf{w}) := \sum_{\substack{\mathbf{x}, \mathbf{x}' \in \mathbb{Z}^2 \\ \mathbf{x} - \mathbf{x}' = \mathbf{w}}} \alpha(\mathbf{x}) \overline{\alpha}(\mathbf{x}').$$

The vectors \mathbf{y} take the form $e\mathbf{u}$ where $\mathbf{u} = (u_1, u_2) \in \mathbb{Z}^2$ is primitive, and where $e \leq E$ is a positive integer. Since $\det \mathbf{M} = 1$, we see that the general solution of the linear equation $\mathbf{w}^T \mathbf{M} \mathbf{u} = 0$ is

$$\mathbf{w} = c \widetilde{\mathbf{M}} \mathbf{u}$$

for $c \in \mathbb{Z}$. Hence

$$T(\alpha) = \sum_{e \leq E} \sum_{\substack{\mathbf{u} \in \mathbb{Z}^2 \\ e \widetilde{\mathbf{M}} \mathbf{u} \in \widetilde{B}}}^* \sum_{c \in \mathbb{Z}} (\alpha * \alpha)(c \widetilde{\mathbf{M}} \mathbf{u}) = \sum_{e \leq E} \sum_{e\mathbf{z} \in \widetilde{B}}^* \sum_{c \in \mathbb{Z}} (\alpha * \alpha)(c\mathbf{z}),$$

where $\sum_{\mathbf{z}}^*$ denotes summation for primitive vectors $\mathbf{z} = (z_1, z_2)$. We note that Cauchy's inequality yields $|(\alpha * \alpha)(\mathbf{w})| \leq \|\alpha\|_2^2$ for any $\mathbf{w} \in \mathbb{Z}^2$. Thus the contribution to the above sum from terms with $c = 0$ is $O(\|\alpha\|_2^2 B^2)$, whence

$$T(\alpha) = T_0(\alpha) + O(\|\alpha\|_2^2 B^2), \quad (7.3)$$

with

$$T_0(\alpha) := \sum_{e \leq E} \sum_{e\mathbf{z} \in \widetilde{B}}^* \sum_{c \in \mathbb{Z} \setminus \{0\}} (\alpha * \alpha)(c\mathbf{z}).$$

We now use the Möbius function to pick out the condition $\gcd(z_1, z_2) = 1$, giving

$$T_0(\alpha) = \sum_{e \leq E} \sum_{c \in \mathbb{Z} \setminus \{0\}} \sum_{b=1}^{\infty} \mu(b) T_0(\alpha; b, c, e),$$

where

$$T_0(\alpha; b, c, e) := \sum_{\substack{ec^{-1}\mathbf{w} \in \widetilde{B} \\ bc|\mathbf{w}}} (\alpha * \alpha)(\mathbf{w}).$$

For $ec^{-1}\mathbf{w} \in \widetilde{B}$ we deduce that

$$(2M_0)^{-1}e^{-1}cB \leq |\mathbf{w}| \leq 4M_0e^{-1}cB. \quad (7.4)$$

Moreover we will have $(\alpha * \alpha)(\mathbf{w}) = 0$ unless $|\mathbf{w}| \leq 2A$. Thus we can restrict the summation over c to the range

$$0 < |c| \leq 4M_0AB^{-1}E = C,$$

say.

We may now handle certain ranges of b and c trivially. To this end we give ourselves a parameter $K \geq 1$ which we will choose in due course. In view of (7.4) the number of available vectors $\mathbf{w} \equiv 0 \pmod{bc}$, corresponding to a particular triple b, c, e is $O(B^2b^{-2}e^{-2})$. Since $|(\alpha * \alpha)(\mathbf{w})| \leq \|\alpha\|_2^2$ we deduce that the contribution to $T_0(\alpha)$ from $b > K$ is

$$\ll \sum_{e \leq E} \sum_{c \leq C} \sum_{b > K} \|\alpha\|_2^2 B^2 b^{-2} e^{-2} \ll \|\alpha\|_2^2 ABEK^{-1}.$$

Similarly the contribution from $|c| \leq CK^{-1}$ is

$$\ll \sum_{e \leq E} \sum_{|c| \leq C/K} \sum_b \|\alpha\|_2^2 B^2 b^{-2} e^{-2} \ll \|\alpha\|_2^2 ABEK^{-1}.$$

It follows that

$$T_0(\alpha) = \sum_{e \leq E} \sum_{b \leq K} \mu(b) \sum_{C/K < |c| \leq C} T_0(\alpha; b, c, e) + O(\|\alpha\|_2^2 ABEK^{-1}).$$

Combining this with (7.2) and (7.3) therefore leads to the conclusion that

$$S(\alpha, \beta; \lambda) \ll \|\beta\|_2 \cdot \|\lambda\|_2 \left\{ \|\alpha\|_2 B + \|\alpha\|_2 (ABEK^{-1})^{1/2} + T_1(\alpha)^{1/2} \right\}$$

for any $A \geq B \geq 1$ and $K \geq 1$, with

$$T_1(\alpha) := \sum_{e \leq E} \sum_{b \leq K} \sum_{C/K < |c| \leq C} |T_0(\alpha; b, c, e)|.$$

We now open up the convolution $\alpha * \alpha$ to obtain

$$T_0(\alpha; b, c, e) = \sum_{\mathbf{x}' \in \mathbb{Z}^2} \bar{\alpha}(\mathbf{x}') \sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{x}' \pmod{bc}}} \alpha(\mathbf{x}),$$

where

$$R = \{\mathbf{x} \in \mathbb{Z}^2 : (2M_0)^{-1} e^{-1} cB \leq |\mathbf{x} - \mathbf{x}'| \leq 4M_0 e^{-1} cB\}.$$

Thus

$$|T_0(\alpha; b, c, e)| \leq \sum_{\mathbf{x}' \in \mathbb{Z}^2} |\alpha(\mathbf{x}')| \max_R \left| \sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{x}' \pmod{bc}}} \alpha(\mathbf{x}) \right|,$$

where R now runs over all squares with sides aligned to the axes. It follows from Cauchy's inequality that

$$\begin{aligned} |T_0(\alpha; b, c, e)|^2 &\leq \|\alpha\|_2^2 \sum_{\mathbf{x}' \in \mathcal{A}} \max_R \left| \sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{x}' \pmod{bc}}} \alpha(\mathbf{x}) \right|^2 \\ &\ll \|\alpha\|_2^2 (1 + A^2 (bc)^{-2}) T_2(\alpha; |bc|), \end{aligned}$$

with

$$T_2(\alpha; q) := \sum_{\mathbf{u} \pmod{q}} \max_R \left| \sum_{\substack{\mathbf{x} \in R \\ \mathbf{x} \equiv \mathbf{u} \pmod{q}}} \alpha(\mathbf{x}) \right|^2. \quad (7.5)$$

If we insist that $E \leq B$ then $1 + A|bc|^{-1} \leq 1 + AKC^{-1} \ll BKE^{-1}$, whence

$$T_1(\alpha) \ll E \sum_{b \leq K} \sum_{C/K < c \leq C} \|\alpha\|_2 BKE^{-1} T_2(\alpha; |bc|)^{1/2}.$$

Hence, if $\tau(q)$ denotes the usual divisor function, and we set

$$T_3 := \sum_{q \leq CK} q^2 T_2(\alpha; q), \quad (7.6)$$

a further application of Cauchy's inequality yields

$$\begin{aligned} T_1(\alpha) &\ll BK\|\alpha\|_2 \left\{ \sum_{C/K < q \leq CK} \tau(q)^2 q^{-2} \right\}^{1/2} T_3^{1/2} \\ &\ll BK\|\alpha\|_2 \{KC^{-1}(\log C)^3\}^{1/2} T_3^{1/2} \\ &\ll (BK)^{3/2} (AE)^{-1/2} (\log A)^{3/2} \|\alpha\|_2 T_3^{1/2}. \end{aligned}$$

We may now conclude as follows.

Lemma 15. — *Let $A \geq B \geq E \geq 1$ and define $C = 4M_0AB^{-1}E$. Let $\alpha, \beta : \mathbb{Z}^2 \rightarrow \mathbb{C}$ be functions supported on the sets (7.1), and let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$. Define $T_2(\alpha; q)$ and T_3 as in (7.5) and (7.6). Then for any $K \geq 1$ we have*

$$S(\alpha, \beta; \lambda) \ll \|\beta\|_2 \cdot \|\lambda\|_2 \left(\|\alpha\|_2 B + \|\alpha\|_2 (ABEK^{-1})^{1/2} + T_1(\alpha)^{1/2} \right),$$

with

$$T_1(\alpha) \ll (BK)^{3/2} (AE)^{-1/2} (\log A)^2 \|\alpha\|_2 T_3^{1/2}.$$

Before proceeding further it may be helpful to comment on the above estimate. For the purposes of this illustration we shall suppose that K is chosen so that

$$\max\{E^2, \log^2 A\} \leq K \leq A/B. \quad (7.7)$$

If we merely estimate $S(\alpha, \beta; \lambda)$ via Cauchy's inequality, using the fact that $\mathbf{x}^T \mathbf{M} \mathbf{y} = l$ has $O(AB \log A)$ solutions \mathbf{x}, \mathbf{y} for each given l , we are led to a trivial bound

$$S(\alpha, \beta; \lambda) \ll \|\alpha\|_2 \cdot \|\beta\|_2 \cdot \|\lambda\|_2 \sqrt{AB \log A}.$$

Hence the first term in the above lemma saves at least $\sqrt{A/B}$. Similarly the second term saves at least $\sqrt{K/E}$. Both these are at least $K^{1/4}$. To analyse the third term we use the argument in (6.4) to deduce that

$$T_2(\alpha; q) \ll A^2 q^{-2} \|\alpha\|_2^2$$

for $q \ll A$. We therefore obtain the trivial bound

$$\begin{aligned} T_1(\alpha) &\ll \|\alpha\|_2^2 A^{1/2} B^{3/2} C^{1/2} K^2 E^{-1/2} \log^2 A \\ &\ll \|\alpha\|_2^2 ABK^2 \log^2 A \\ &\ll \|\alpha\|_2^2 ABK^3. \end{aligned}$$

For comparison, in order for the third term in Lemma 15 to produce a comparable saving to that in the first two terms, one would wish to replace the above by

$$T_1(\alpha) \ll \|\alpha\|_2^2 ABK^{-1},$$

say. The type of saving we require is exactly that given by Lemma 14, providing that we work with $S(\alpha_0, \beta; \lambda)$.

We now show how Lemma 15 can be applied to estimate

$$\mathcal{E}(G, H, V) = \sum_{x \in \mathfrak{o}_L} \sum_{y \in \mathfrak{o}_L} \alpha_0(x) \beta(y) \lambda(\widetilde{\text{Tr}}(x, y)),$$

when $G = \log V$. We have yet to specify the parameter Q used in the definition of $\hat{\alpha}$, but we shall do this shortly. Under suitable hypotheses we will see in § 8 and § 9 that the main term $\mathcal{M}(G, H, V)$ has order at least $(\log V)^{1-3n} H^n V^{2n}$. Our goal is to show that $\mathcal{E}(G, H, V)$ is smaller than this by at least a power of H .

If $x = x_1 + x_2\tau$ and $y = y_1 + y_2\tau$, then $\widetilde{\text{Tr}}(x, y) = \mathbf{x}^T \mathbf{M} \mathbf{y}$, where

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By abuse of notation we will write $\alpha_0(\mathbf{x}) = \alpha_0(x)$ and $\beta(\mathbf{y}) = \beta(y)$. From our definitions of $\alpha, \hat{\alpha}$ and ω in (3.15), (4.13) and (5.8) we see that if $\alpha_0(x) \neq 0$ then $x = \delta \mathbf{N}_{K/L}(\mathbf{u})$ for some $\mathbf{u} \in \mathcal{U}$. Moreover, from (3.14) we deduce that $\delta \mathbf{N}_{K/L}(\mathbf{u}) = U^{n/2} \delta \mathbf{N}_{K/L}(\mathbf{u}^{(\mathbb{R})}) + o(U^{n/2})$ as $G \rightarrow \infty$. Since $\delta \mathbf{N}_{K/L}(\mathbf{u}^{(\mathbb{R})}) \neq 0$ it follows that $\alpha(\mathbf{x})$ is supported on a suitable set $|\mathbf{x}| \leq A$ with $A = c_1 U^{n/2}$ for a certain constant $c_1 > 0$. We may analyse the support of β similarly. Since $\mathbf{N}_{K/L}(\mathbf{v}^{(\mathbb{R})}) \neq 0$ we deduce that $\beta(\mathbf{y})$ is supported on a set $B \leq |\mathbf{y}| \leq 2B$ with $B = c_2 V^{n/2}$ for a suitable positive constant c_2 . Moreover, $\beta(\mathbf{y})$ is also supported on vectors \mathbf{y} with $\gcd(y_1, y_2) \ll 1$, by (3.13). We may therefore take E of order 1.

Estimates for $\|\beta\|_2$ and $\|\lambda\|_2$ are given by Lemma 4, while Lemma 11 handles $\|\alpha_0\|_2$. Inserting these bounds into Lemma 15 and adopting the assumption (7.7), together with the bounds $Q^{n+5} \leq U^{1-\eta}$ and $G \leq U^{1/(n+1)}$, we then see that

$$B + (ABEK^{-1})^{1/2} \ll U^{n/4} V^{n/4} K^{-1/2}$$

and

$$(BK)^{3/2} (AE)^{-1/2} (\log A)^2 \ll U^{-n/4} V^{3n/4} K^{5/2},$$

so that

$$\begin{aligned} \mathcal{E}(G, H, V) &\ll_{\eta} V^{(n+\eta)/2} W^{(n+\eta)/2} \left(U^{(n+\eta)/2} \cdot U^{n/4} V^{n/4} K^{-1/2} \right. \\ &\quad \left. + U^{-n/8} V^{3n/8} K^{5/4} \cdot U^{(n+\eta)/4} T_3^{1/4} \right) G^{O(1)}. \end{aligned}$$

We now apply Lemma 14, along with the bounds $C \ll AB^{-1}E$ and $G = O_{\eta}(V^{\eta})$, to deduce that

$$T_3 = \sum_{q \leq CK} q^2 T_2(\alpha; q) \ll_{\eta} CKQ^{-1} U^{2n} V^{O(\eta)} \ll_{\eta} KQ^{-1} U^{5n/2} V^{-n/2+O(\eta)},$$

providing that

$$Q \leq CK \leq U^{1/(n+16)}. \quad (7.8)$$

Note that when $G = \log V$ the condition $G \leq U^{1/(n+1)}$ is automatic for large V . Moreover, if $Q \leq U^{1/(n+16)}$ then we automatically have $Q^{n+5} \leq U^{1-\eta}$, if η is small enough.

Thus, on recalling that $U = HV$ and $W = H^{1/2}V$, we deduce that

$$\begin{aligned} \mathcal{E}(G, H, V) &\ll_{\eta} H^{n/4} V^{n+O(\eta)} \left(H^{3n/4} V^n K^{-1/2} + H^{3n/4} V^n K^{3/2} Q^{-1/4} \right) \\ &= H^n V^{2n+O(\eta)} (K^{-1/2} + K^{3/2} Q^{-1/4}). \end{aligned}$$

It is now apparent that we should take $K = Q^{1/8}$, which will satisfy (7.7) and (7.8) if $Q \ll H^{4n/7}$ and $(\log V)^{16} \ll Q \ll H^{-4n} U^{8/(n+16)}$. We conclude as follows.

Lemma 16. — *Let $G = \log V$ and assume that*

$$(\log V)^{16} \ll Q \ll \min\{H^{n/2}, H^{-5n}U^{8/(n+16)}\}.$$

Then we have

$$\mathcal{E}(G, H, V) \ll_{\eta} Q^{-1/16} H^n V^{2n+O(\eta)}.$$

8. Estimation of the main term

The purpose of this section and the next is to produce a satisfactory estimate for the sum

$$\mathcal{M} = \mathcal{M}(G, H, V) = \sum_{x \in \mathfrak{o}_L} \sum_{y \in \mathfrak{o}_L} \hat{\alpha}(x) \beta(y) \lambda(\widetilde{\text{Tr}}(x, y)),$$

as $G \rightarrow \infty$, where $\hat{\alpha}$ is the approximation to α which we constructed in (4.13). Our goal will be to demonstrate that \mathcal{M} has order $G^{-2n} H^n V^{2n}$, which simple heuristics suggest to be the expected size of $\mathcal{N}(G, H, V)$. In fact we will fall a little short of this, showing that $\mathcal{M} \gg G^{1-3n} H^n V^{2n}$ when suitable constraints are placed on the parameters Q, G, H and V .

Let

$$\begin{aligned} \mathcal{V}_1 &:= \{\mathbf{v} \in \mathcal{V} \cap \mathbb{Z}^n : \mathbf{v} \equiv \mathbf{v}^{(M)} \pmod{M}, \text{ (3.12) holds}\}, \\ \mathcal{W}_1 &:= \{\mathbf{w} \in \mathcal{W} \cap \mathbb{Z}^n : \mathbf{w} \equiv \mathbf{w}^{(M)} \pmod{M}\}, \end{aligned} \tag{8.1}$$

where \mathcal{V}, \mathcal{W} are given by (3.14). Opening up the functions β and λ from (3.16) and (3.17), respectively, we find that

$$\mathcal{M} = \sum_{\mathbf{v} \in \mathcal{V}_1} \sum_{\mathbf{w} \in \mathcal{W}_1} \sum_{\substack{x \in \mathfrak{o}_L \\ \text{Tr}_{L/\mathbb{Q}}(x \mathbf{N}_{K/L}(\mathbf{v})) = 2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w})}} \hat{\alpha}(x) = \sum_{\mathbf{v} \in \mathcal{V}_1} \sum_{\mathbf{w} \in \mathcal{W}_1} \mathcal{M}(\mathbf{v}, \mathbf{w}),$$

say. Recall that $D_L = \tau - \tau^\sigma$ where $\{1, \tau\}$ is a \mathbb{Z} -basis for \mathfrak{o}_L . It will be convenient to make the observation that

$$2 \text{Tr}_{L/\mathbb{Q}}(\tau^2) - \text{Tr}_{L/\mathbb{Q}}(\tau)^2 = D_L^2,$$

which is a non-zero integer. By enlarging the weak approximation set S in Theorem 2, if necessary, we may clearly assume that M contains any prime divisors of $2D_L^2$.

We now set in motion our analysis of $\mathcal{M}(\mathbf{v}, \mathbf{w})$, for given $\mathbf{v} \in \mathcal{V}_1$ and $\mathbf{w} \in \mathcal{W}_1$. Suppose that $\mathbf{N}_{K/L}(\mathbf{v})$ decomposes as $\mathbf{N}_1(\mathbf{v}) + \mathbf{N}_2(\mathbf{v})\tau$, for suitable forms $\mathbf{N}_1, \mathbf{N}_2$ of degree $n/2$. Then (3.12) demands that $\mathbf{N}_1(\mathbf{v})$ and $\mathbf{N}_2(\mathbf{v})$ be coprime. Moreover, in view of (3.8) and our convention that $\omega_1 = 1$ in the integral basis $\{\omega_1, \dots, \omega_n\}$ for K over \mathbb{Q} , it is clear that

$$\mathbf{N}_1(\mathbf{v}) \equiv 1 \pmod{M}, \quad \mathbf{N}_2(\mathbf{v}) \equiv 0 \pmod{M}.$$

Let us write $x = x_1 + x_2\tau$. The constraint in the x summation is equivalent to

$$a_1 x_1 + a_2 x_2 = b,$$

for integers a_1, a_2, b such that

$$\begin{aligned} a_1 &= \text{Tr}_{L/\mathbb{Q}}(\mathbf{N}_{K/L}(\mathbf{v})) = 2\mathbf{N}_1(\mathbf{v}) + \text{Tr}_{L/\mathbb{Q}}(\tau)\mathbf{N}_2(\mathbf{v}), \\ a_2 &= \text{Tr}_{L/\mathbb{Q}}(\tau \mathbf{N}_{K/L}(\mathbf{v})) = \text{Tr}_{L/\mathbb{Q}}(\tau)\mathbf{N}_1(\mathbf{v}) + \text{Tr}_{L/\mathbb{Q}}(\tau^2)\mathbf{N}_2(\mathbf{v}), \\ b &= 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}). \end{aligned} \tag{8.2}$$

In order to analyse $\mathcal{M}(\mathbf{v}, \mathbf{w})$ we will need to recall the expression for $\hat{\alpha}(x)$. Let the function $\omega(x) = \omega(x_1, x_2)$ be given by (5.8), with key properties as in Lemma 9. Then for any parameter $Q \geq 1$ and $x \in \mathfrak{o}_L$ we have

$$\hat{\alpha}(x) = \omega(x) \sum_{q \leq Q} \sum_{t \pmod{q}}^* e_q^{(L)}(-tx) \sum_{z \pmod{q}} \rho(z, q) e_q^{(L)}(tz),$$

where $\rho(z, q)$ is given by (5.6) and the notation \sum^* means that the sum is taken over $t = t_1 + t_2\tau$, with $t_1, t_2 \in \mathbb{Z}/q\mathbb{Z}$ such that $\gcd(q, t_1, t_2) = 1$. Define

$$c_q(t) := \sum_{z \pmod{q}} \rho(z, q) e_q^{(L)}(tz), \quad (8.3)$$

for $t \pmod{q}$. It is clear from (6.3) that $|c_q(t)| \leq 1$. Inserting our expression for $\hat{\alpha}(x)$ and breaking the inner sum into congruence classes modulo q , we see that

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \sum_{q \leq Q} \sum_{t \pmod{q}}^* c_q(t) \sum_{r \pmod{q}} e_q^{(L)}(-tr) \mathcal{L}_r(\mathbf{v}, \mathbf{w}),$$

where

$$\mathcal{L}_r(\mathbf{v}, \mathbf{w}) := \sum_{\substack{a_1 x_1 + a_2 x_2 = b \\ x \equiv r \pmod{q}}} \omega(x).$$

We proceed to investigate the summation conditions on x . We first prove that a_1 and a_2 are not both zero, that b is non-zero, and that

$$\gcd(a_1, a_2) = 2^\kappa, \quad \kappa := \begin{cases} 1, & \text{if } 2 \mid \text{Tr}_{L/\mathbb{Q}}(\tau), \\ 0, & \text{if } 2 \nmid \text{Tr}_{L/\mathbb{Q}}(\tau). \end{cases} \quad (8.4)$$

In particular $\gcd(a_1, a_2) \mid b$.

To establish the claim we observe that $\gcd(a_1, a_2)$ is a common divisor of

$$2a_2 - \text{Tr}_{L/\mathbb{Q}}(\tau)a_1 = D_L^2 \mathbf{N}_2(\mathbf{v})$$

and

$$\text{Tr}_{L/\mathbb{Q}}(\tau^2)a_1 - \text{Tr}_{L/\mathbb{Q}}(\tau)a_2 = D_L^2 \mathbf{N}_1(\mathbf{v}).$$

However \mathbf{v} and \mathbf{w} are in the regions (3.14) with $\mathbf{N}_{K/L}(\mathbf{v}^{(\mathbb{R})})$ and $\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}^{(\mathbb{R})})$ non-zero. Thus if G is large enough we will have $\mathbf{N}_{K/L}(\mathbf{v})$ and $\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w})$ non-zero. Hence b will be non-zero and similarly, since $D_L \neq 0$, the numbers a_1 and a_2 cannot both be zero. We also see that

$$\gcd(a_1, a_2) \mid D_L^2 \gcd(\mathbf{N}_1(\mathbf{v}), \mathbf{N}_2(\mathbf{v})) = D_L^2.$$

Since $\mathbf{N}_1(\mathbf{v}) \equiv 1 \pmod{p}$ and $\mathbf{N}_2(\mathbf{v}) \equiv 0 \pmod{p}$ for any prime divisor p of $2D_L^2$, the claim readily follows.

We now write $a'_i = 2^{-\kappa} a_i$ for $i = 1, 2$, and $b' = 2^{-\kappa} b$, so that the first condition on x becomes $a'_1 x_1 + a'_2 x_2 = b'$. Thus the sum will be empty unless $a'_1 r_1 + a'_2 r_2 \equiv b' \pmod{q}$, as we now assume. For future use we note that this is equivalent to demanding that

$$2^{-\kappa} \text{Tr}_{L/\mathbb{Q}}(r \mathbf{N}_{K/L}(\mathbf{v})) \equiv 2^{1-\kappa} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) \pmod{q}. \quad (8.5)$$

We may therefore write $a'_1 r_1 + a'_2 r_2 = b' + qk$ for some $k \in \mathbb{Z}$. Then, if we take $x = r + qy$, the summation conditions on x translate into the requirement that $a'_1 y_1 + a'_2 y_2 = -k$. Since a'_1 is coprime to a'_2 we can find integers $\overline{a}_1, \overline{a}_2$ such that $a'_1 \overline{a}_1 + a'_2 \overline{a}_2 = 1$. We then have $a'_1(y_1 + k\overline{a}_1) + a'_2(y_2 + k\overline{a}_2) = 0$, so that $y_1 + k\overline{a}_1 = -ma'_2$ and $y_2 + k\overline{a}_2 = ma'_1$ for some integer

m . It follows that the summation conditions $a_1x_1 + a_2x_2 = b$ and $x \equiv r \pmod{q}$ are satisfied if and only if x_1 and x_2 take the forms $x_1 = r_1 - \overline{a_1}kq - a'_2qm$ and $x_2 = r_2 - \overline{a_2}kq + a'_1qm$ respectively. Our conclusion is therefore that

$$\mathcal{L}_r(\mathbf{v}, \mathbf{w}) = \sum_{m \in \mathbb{Z}} f(m),$$

if (8.5) holds, where

$$f(m) := \omega(r_1 - \overline{a_1}kq - a'_2qm, r_2 - \overline{a_2}kq + a'_1qm).$$

We would now like to replace the discrete summation over m by a continuous integral. For this a relatively crude approach is available to us through Lemma 9. Thus if $v \in [0, 1]$ it follows that

$$f(m+v) - f(m) \ll U^{-n/2}q \max\{|a'_1|, |a'_2|\}.$$

Moreover Lemma 9 tells us that ω is supported on a disc of radius $O(U^{n/2})$, whence f is supported on an interval with length $O(U^{n/2}/(q \max\{|a'_1|, |a'_2|\}))$. Recalling that $U \geq V$ in Lemma 3 and $\max\{|a'_1|, |a'_2|\} \ll V^{n/2}$, this therefore produces the conclusion

$$\left| \int_{-\infty}^{\infty} f(m) dm - \sum_{m \in \mathbb{Z}} f(m) \right| \ll 1 + qU^{-n/2} \max\{|a'_1|, |a'_2|\} \ll q.$$

Assuming that $a_2 \neq 0$ the change of variables $m = q^{-1}(-2^\kappa x + r_1/a'_2 - \overline{a_1}kq/a'_2)$ now yields

$$\mathcal{L}_r(\mathbf{v}, \mathbf{w}) = \frac{2^\kappa}{q} I(\mathbf{v}, \mathbf{w}) + O(q),$$

for r satisfying (8.5), with

$$I(\mathbf{v}, \mathbf{w}) := \int_{-\infty}^{\infty} \omega(a_2x, -a_1x + b/a_2) dx. \quad (8.6)$$

Here a_1, a_2, b depend on \mathbf{v} and \mathbf{w} and are given by (8.2). If $a_2 = 0$ we reverse the rôles of a_1 and a_2 to produce an integral involving $\omega(-a_2x + b/a_1, a_1x)$.

Recall the estimate $|c_q(t)| \leq 1$ that we recorded above. Inserting our estimate for $\mathcal{L}_r(\mathbf{v}, \mathbf{w})$ into that for $\mathcal{M}(\mathbf{v}, \mathbf{w})$, it now follows that

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = 2^\kappa I(\mathbf{v}, \mathbf{w}) \sum_{q \leq Q} \frac{1}{q} \sum_{t \pmod{q}}^* c_q(t) \sum_{\substack{r \pmod{q} \\ (8.5) \text{ holds}}} e_q^{(L)}(-tr) + O(Q^6).$$

We now sum both sides over all $\mathbf{v} \in \mathcal{V}_1$ and $\mathbf{w} \in \mathcal{W}_1$. On observing that $\#\mathcal{V}_1 = O(G^{-n}V^n)$ and $\#\mathcal{W}_1 = O(G^{-n}W^n)$, we see that the overall contribution from the error term is

$$\ll Q^6 G^{-2n} V^n W^n \ll Q^6 H^{n/2} V^{2n}.$$

This will be satisfactory if Q is sufficiently small compared to H . Our work so far has shown that

$$\mathcal{M} = 2^\kappa \sum_{\mathbf{v} \in \mathcal{V}_1} \sum_{\mathbf{w} \in \mathcal{W}_1} I(\mathbf{v}, \mathbf{w}) \sum_{q \leq Q} \frac{1}{q} \mathcal{C} + O(Q^6 H^{n/2} V^{2n}), \quad (8.7)$$

where $\mathcal{V}_1, \mathcal{W}_1$ are given by (8.1), and

$$\mathcal{C} := \sum_{t \pmod{q}}^* c_q(t) \sum_{\substack{r \pmod{q} \\ (8.5) \text{ holds}}} e_q^{(L)}(-tr).$$

Opening up (8.3), we find that

$$\mathcal{C} = \sum_{\substack{r \pmod{q} \\ (8.5) \text{ holds}}} \sum_{z \pmod{q}} \rho(z, q) \sum_{t \pmod{q}}^* e_q^{(L)}(t(z-r)).$$

We have seen that the condition (8.5) can be written $a'_1 r_1 + a'_2 r_2 \equiv b' \pmod{q}$, for non-zero integers a'_1, a'_2, b' such that $\gcd(a'_1, a'_2) = 1$. The inner sum is a Ramanujan sum and thus it follows from (4.12), combined with (4.9), that

$$\begin{aligned} \mathcal{C} &= \sum_{u|q} u^2 \mu(q/u) \sum_{\substack{r \pmod{q} \\ a'_1 r_1 + a'_2 r_2 \equiv b' \pmod{q}}} \sum_{\substack{z \pmod{q} \\ z \equiv r \pmod{u}}} \rho(z, q) \\ &= \sum_{u|q} u^2 \mu(q/u) \sum_{\substack{r \pmod{q} \\ a'_1 r_1 + a'_2 r_2 \equiv b' \pmod{q}}} \rho(r, u). \end{aligned}$$

Let $q = uv$. Writing $r = r' + ur''$ for $r' \pmod{u}$ and $r'' \pmod{v}$ we see that the sum over r is equal to

$$\sum_{\substack{r' \pmod{u} \\ a'_1 r'_1 + a'_2 r'_2 \equiv b' \pmod{u}}} \rho(r', u) \# \{r'' \pmod{v} : a'_1 r''_1 + a'_2 r''_2 \equiv g(r') \pmod{v}\},$$

where $g(r') = u^{-1}(b' - a'_1 r'_1 - a'_2 r'_2)$. Since a'_1 and a'_2 are coprime it follows that there are precisely v possibilities for r'' . The definition (5.6) of $\rho(r', u)$ therefore reveals that

$$\mathcal{C} = M^n q \sum_{u|q} \frac{u \mu(q/u)}{[M, u]^n} \# \left\{ \mathbf{s} \pmod{[M, u]} : \begin{array}{l} \mathbf{s} \equiv \mathbf{u}^{(M)} \pmod{M}, \\ 2^{-\kappa} F(\mathbf{v}; \mathbf{w}; \mathbf{s}) \equiv 0 \pmod{u} \end{array} \right\},$$

where

$$F(\mathbf{v}; \mathbf{w}; \mathbf{s}) := \text{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{s}) \mathbf{N}_{K/L}(\mathbf{v})) - 2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}).$$

This is the polynomial that underpins the variety introduced in (3.5).

We proceed to insert this expression for \mathcal{C} into (8.7). We will use the Möbius function to remove the coprimality condition (3.12), which is implicit in the definition (8.1) of \mathcal{V}_1 . We therefore arrive at the estimate

$$\begin{aligned} \mathcal{M} &= 2^\kappa M^n \sum_{k=1}^{\infty} \mu(k) \sum_{q \leq Q} \sum_{u|q} \frac{u \mu(q/u)}{[M, u]^n} \sum_{\substack{\mathbf{s} \pmod{[M, u]} \\ \mathbf{s} \equiv \mathbf{u}^{(M)} \pmod{M}}} \\ &\quad \times \sum_{\substack{(\mathbf{v}, \mathbf{w}) \in \mathcal{V}_2 \times \mathcal{V}_1 \\ 2^{-\kappa} F(\mathbf{v}; \mathbf{w}; \mathbf{s}) \equiv 0 \pmod{u}}} I(\mathbf{v}, \mathbf{w}) + O\left(Q^6 H^{n/2} V^{2n}\right), \end{aligned} \tag{8.8}$$

where $I(\mathbf{v}, \mathbf{w})$ is given by (8.6) and \mathcal{V}_2 is defined as for \mathcal{V}_1 , but with the condition (3.12) replaced by $k \mid \mathbf{N}_{K/L}(\mathbf{v})$. This latter condition is taken componentwise as $k \mid \mathbf{N}_1(\mathbf{v})$ and $k \mid \mathbf{N}_2(\mathbf{v})$, in the usual way.

The sum over k is empty when $k \gg V^{n/2}$ so that we only need consider values $k \ll V^{n/2}$. However we need to reduce this range further. Since $\mathbf{N}_{K/L}(\mathbf{v}^{(\mathbb{R})}) \neq 0$ it follows from (8.2) and the definition of \mathcal{V} that

$$\max\{|a_1|, |a_2|\} \gg V^{n/2}.$$

The integration in (8.6) is therefore over an interval of length $\ll V^{-n/2}U^{n/2} = H^{n/2}$, by Lemma 9. A second application of Lemma 9 to bound the size of ω now yields

$$I(\mathbf{v}, \mathbf{w}) \ll H^{n/2}. \quad (8.9)$$

Hence there is an absolute constant $c > 0$ such that the overall contribution to \mathcal{M} from terms with $k > K$ is

$$\begin{aligned} &\ll H^{n/2} \sum_{K < k \ll V^{n/2}} \sum_{q \leq Q} \sum_{u|q} u \sum_{\mathbf{w} \in \mathcal{W}_1} M_k(cV) \\ &\ll_{\eta} Q^{2+\eta} H^{n/2} W^n \sum_{K < k \ll V^{n/2}} M_k(cV), \end{aligned}$$

where

$$M_k(X) := \#\{\mathbf{v} \in \mathbb{Z}^n : |\mathbf{v}| \leq X, k \mid \mathbf{N}_{K/L}(\mathbf{v})\}.$$

Clearly $k^2 \mid \mathbf{N}_{K/\mathbb{Q}}(\mathbf{v})$ whenever $k \mid \mathbf{N}_{K/L}(\mathbf{v})$, whence

$$M_k(X) \leq \sum_{\substack{N \ll X^n \\ k^2 \mid N}} \#\{\mathbf{v} \in \mathbb{Z}^n : |\mathbf{v}| \leq X, \mathbf{N}_{K/\mathbb{Q}}(\mathbf{v}) = N\} \ll_{\eta} \left(1 + \frac{X^n}{k^2}\right) X^{\eta}.$$

The overall contribution to the main term from $k > K$ is therefore seen to be

$$\begin{aligned} &\ll_{\eta} Q^{2+\eta} H^{n/2} W^n V^{n+\eta} \sum_{K < k \ll V^{n/2}} \frac{1}{k^2} \\ &\ll_{\eta} \frac{Q^{2+\eta} H^n V^{2n+\eta}}{K}. \end{aligned}$$

This allows us to truncate the summation over k in (8.8) with acceptable error.

We now replace the sum over \mathbf{s} in (8.8) by one in which the variable runs modulo $[M, u, k]$. This has the effect of multiplying it by $[M, u, k]^{-n} [M, u]^n$. We may therefore summarise our findings in the following result.

Lemma 17. — *Let $K \geq 1$ and let κ be given by (8.4). Then we have*

$$\begin{aligned} \mathcal{M} &= 2^{\kappa} M^n \sum_{k \leq K} \mu(k) \sum_{q \leq Q} \sum_{u|q} \frac{u \mu(q/u)}{\Delta^n} \sum_{\substack{\mathbf{s} \pmod{\Delta} \\ \mathbf{s} \equiv \mathbf{u}^{(M)} \pmod{M}}} \mathcal{K}_{k,u}(\mathbf{s}) \\ &\quad + O_{\eta} \left(H^n V^{2n+\eta} \left\{ \frac{Q^6}{H^{n/2}} + \frac{Q^{2+\eta}}{K} \right\} \right), \end{aligned}$$

where $\Delta := [M, u, k]$ and

$$\mathcal{K}_{k,u}(\mathbf{s}) := \sum_{\substack{(\mathbf{v}, \mathbf{w}) \in \mathcal{V}_2 \times \mathcal{W}_1 \\ 2^{-\kappa} F(\mathbf{v}; \mathbf{w}; \mathbf{s}) \equiv 0 \pmod{u}}} I(\mathbf{v}, \mathbf{w}).$$

Here \mathcal{W}_1 is given by (8.1), I is given by (8.6) and \mathcal{V}_2 is defined as for \mathcal{V}_1 in (8.1), but with (3.12) replaced by $k \mid \mathbf{N}_{K/L}(\mathbf{v})$.

The next phase of the argument concerns a detailed analysis of $\mathcal{K}_{k,u}(\mathbf{s})$. It is natural to break the sum over \mathbf{v} and \mathbf{w} into congruence classes modulo $\Delta = [M, u, k]$. Let

$$\mathcal{J}(\Delta) = \mathcal{J}(\mathbf{p}, \mathbf{q}, \mathbf{s}; \Delta) := \sum_{\substack{\mathbf{w} \in \mathcal{W} \cap \mathbb{Z}^n \\ \mathbf{w} \equiv \mathbf{q} \pmod{\Delta}}} \sum_{\substack{\mathbf{v} \in \mathcal{V} \cap \mathbb{Z}^n \\ \mathbf{v} \equiv \mathbf{p} \pmod{\Delta}}} I(\mathbf{v}, \mathbf{w}).$$

Then we have

$$\mathcal{K}_{k,u}(\mathbf{s}) = \sum_{\mathbf{p}, \mathbf{q}} \mathcal{J}(\Delta), \quad (8.10)$$

where the sum is over $(\mathbf{p}, \mathbf{q}) \in (\mathbb{Z}/\Delta\mathbb{Z})^{2n}$ for which

$$(\mathbf{p}, \mathbf{q}, \mathbf{s}) \equiv (\mathbf{v}^{(M)}, \mathbf{w}^{(M)}, \mathbf{u}^{(M)}) \pmod{M} \quad (8.11)$$

and

$$2^{-\kappa} F(\mathbf{p}; \mathbf{q}; \mathbf{s}) \equiv 0 \pmod{u}, \quad k \mid \mathbf{N}_{K/L}(\mathbf{p}). \quad (8.12)$$

Perhaps the most obvious way to deal with $\mathcal{J}(\Delta)$ is to approximate the sums over \mathbf{w} and \mathbf{v} by integrals. A simple change of variables would then permit us to extract the dependence of $\mathcal{J}(\Delta)$ on $\mathbf{p}, \mathbf{q}, \mathbf{s}$ and Δ . Instead of this it turns out that we can manage with a relatively crude direct comparison of $\mathcal{J}(\Delta)$ with $\mathcal{J}(1)$, as we proceed to show.

It follows from our conditions of summation that $\Delta \ll ku \leq KQ$. It will be convenient to make the additional hypothesis

$$KQ \leq V, \quad (8.13)$$

which implies in particular that $\Delta \ll V$. In view of (8.6) we have

$$\mathcal{J}(\Delta) = \int_{-\infty}^{\infty} \sum_{\substack{\mathbf{w} \in \mathcal{W} \cap \mathbb{Z}^n \\ \mathbf{w} \equiv \mathbf{q} \pmod{\Delta}}} \sum_{\substack{\mathbf{v} \in \mathcal{V} \cap \mathbb{Z}^n \\ \mathbf{v} \equiv \mathbf{p} \pmod{\Delta}}} f(\mathbf{v}, \mathbf{w}) dx,$$

where if $a_i = a_i(\mathbf{v})$ and $b = b(\mathbf{w})$ are given by (8.2) then

$$f(\mathbf{v}, \mathbf{w}) := \omega \left(a_2(\mathbf{v})x, -a_1(\mathbf{v})x + \frac{b(\mathbf{w})}{a_2(\mathbf{v})} \right).$$

We will denote by $T(\mathbf{p}, \mathbf{q}; \Delta)$ the integrand that appears in this expression for $\mathcal{J}(\Delta)$. Using an approach based on the proof of Lemma 10 in § 5 we compare $T(\mathbf{p}, \mathbf{q}; \Delta)$ with $T(\mathbf{0}, \mathbf{0}; \Delta)$. For this we will assume without loss of generality that $\max_{i=1,2} |a_i(\mathbf{v})| = |a_2(\mathbf{v})|$ in the sum over \mathbf{v} . The alternative possibility is accommodated by a simple change of variables in the integral over x . Hence the definition of \mathcal{V} ensures that $a_2(\mathbf{v})$ has order of magnitude $V^{n/2}$ and by Lemma 9 we have $x \ll H^{n/2}$ in $\mathcal{J}(\Delta)$.

Under the change of variables $\mathbf{w} = \mathbf{w}' + \mathbf{q}$ we see that

$$\left| \frac{b(\mathbf{w})}{a_2(\mathbf{v})} - \frac{b(\mathbf{w}')}{a_2(\mathbf{v})} \right| \ll \frac{\Delta |\mathbf{w}'|^{n-1}}{|a_2(\mathbf{v})|} \ll \frac{\Delta W^{n-1}}{V^{n/2}},$$

since $|\mathbf{w}'| \ll \Delta + W \ll W$ by (8.13). We may therefore conclude from Lemma 9 that

$$T(\mathbf{p}, \mathbf{q}; \Delta) = \sum_{\substack{\mathbf{v} \in \mathcal{V} \cap \mathbb{Z}^n \\ \mathbf{v} \equiv \mathbf{p} \pmod{\Delta}}} \sum_{\substack{\mathbf{w}' + \mathbf{q} \in \mathcal{W} \cap \mathbb{Z}^n \\ \mathbf{w}' \equiv \mathbf{0} \pmod{\Delta}}} \left\{ f(\mathbf{v}, \mathbf{w}') + O \left(\frac{\Delta W^{n-1}}{U^{n/2} V^{n/2}} \right) \right\}.$$

Note that the number of \mathbf{v} appearing in the outer sum is $O(\Delta^{-n} V^n)$. For \mathbf{w}' such that $\mathbf{w}' + \mathbf{q} \in \mathcal{W}$ it is clear that $\mathbf{w}' \in \mathcal{W}$ unless \mathbf{w}' is within a distance Δ of the boundary of \mathcal{W} .

Invoking Lemma 9 to deduce that f is bounded absolutely, and recalling that $U = HV$ and $W = H^{1/2}V$, it easily follows that

$$\begin{aligned} |T(\mathbf{p}, \mathbf{q}; \Delta) - T(\mathbf{p}, \mathbf{0}; \Delta)| &\ll (\Delta^{-1}V)^n (\Delta^{-1}W)^{n-1} + \frac{\Delta^{-2n+1}V^{n/2}W^{2n-1}}{U^{n/2}} \\ &\ll \Delta^{-2n+1}H^{(n-1)/2}V^{2n-1}, \end{aligned}$$

by (8.13).

We now repeat the above process by considering the effect of a change of variables $\mathbf{v} = \mathbf{v}' + \mathbf{p}$ in $T(\mathbf{p}, \mathbf{0}; \Delta)$. Recalling that $\Delta \ll KQ \leq V$, by (8.13), we obtain

$$|a_i(\mathbf{v}) - a_i(\mathbf{v}')| \ll \Delta |\mathbf{v}'|^{n/2-1} \ll \Delta V^{n/2-1},$$

for $i = 1, 2$. In particular it follows that

$$\begin{aligned} \left| \frac{b(\mathbf{w})}{a_2(\mathbf{v})} - \frac{b(\mathbf{w})}{a_2(\mathbf{v}')} \right| &\ll \frac{\Delta V^{n/2-1}W^n}{|a_2(\mathbf{v})a_2(\mathbf{v}')|} \\ &\ll \frac{\Delta V^{n/2-1}W^n}{V^n} \\ &= \Delta H^{n/2}V^{n/2-1}, \end{aligned}$$

for any $\mathbf{w} \in \mathscr{W}$. An application of Lemma 9 reveals that

$$T(\mathbf{p}, \mathbf{0}; \Delta) = \sum_{\substack{\mathbf{w} \in \mathscr{W} \cap \mathbb{Z}^n \\ \mathbf{w} \equiv \mathbf{0} \pmod{\Delta}}} \sum_{\substack{\mathbf{v}' + \mathbf{p} \in \mathscr{V} \cap \mathbb{Z}^n \\ \mathbf{v}' \equiv \mathbf{0} \pmod{\Delta}}} \left\{ f(\mathbf{v}', \mathbf{w}) + O\left(\frac{\Delta H^{n/2}V^{n/2-1}}{U^{n/2}}\right) \right\}.$$

To control the error term we note that the total number of available \mathbf{v}', \mathbf{w} in the sums is

$$\ll \frac{W^n}{\Delta^n} \cdot \left(\frac{\Delta + V}{\Delta} + 1 \right)^n \ll \Delta^{-2n}V^nW^n.$$

We conclude that

$$\begin{aligned} |T(\mathbf{p}, \mathbf{0}; \Delta) - T(\mathbf{0}, \mathbf{0}; \Delta)| &\ll (\Delta^{-1}W)^n (\Delta^{-1}V)^{n-1} + \frac{\Delta^{-2n+1}H^{n/2}V^{3n/2-1}W^n}{U^{n/2}} \\ &\ll \Delta^{-2n+1}H^{n/2}V^{2n-1}, \end{aligned}$$

whence

$$T(\mathbf{p}, \mathbf{q}; \Delta) = T(\mathbf{0}, \mathbf{0}; \Delta) + O\left(\Delta^{-2n+1}H^{n/2}V^{2n-1}\right).$$

Summing the latter estimate over \mathbf{p}, \mathbf{q} modulo Δ gives

$$T(\mathbf{0}, \mathbf{0}; 1) = \Delta^{2n}T(\mathbf{0}, \mathbf{0}; \Delta) + O\left(\Delta H^{n/2}V^{2n-1}\right),$$

which we substitute back in to get

$$T(\mathbf{p}, \mathbf{q}; \Delta) = \Delta^{-2n}T(\mathbf{0}, \mathbf{0}; 1) + O\left(\Delta^{-2n+1}H^{n/2}V^{2n-1}\right).$$

Now the outer integral in our expression for $\mathscr{J}(\Delta)$ is over an interval of length $O(H^{n/2})$. We have therefore arrived at the following result.

Lemma 18. — *Assume that $KQ \leq V$. Then we have*

$$\mathscr{J}(\Delta) = \Delta^{-2n}\mathscr{J}(1) + O(\Delta^{-2n+1}H^nV^{2n-1}).$$

This result will allow us to separate out what is in effect the “singular integral” associated to our counting problem. In the notation of (8.6) it is given by

$$\sigma_\infty(G, H, V) := \mathcal{J}(1) = \sum_{\mathbf{w} \in \mathcal{W} \cap \mathbb{Z}^n} \sum_{\mathbf{v} \in \mathcal{V} \cap \mathbb{Z}^n} I(\mathbf{v}, \mathbf{w}). \quad (8.14)$$

It follows from (8.9) that

$$\sigma_\infty(G, H, V) \ll (G^{-1}V)^n \cdot (G^{-1}W)^n \cdot H^{n/2} = G^{-2n} H^n V^{2n}. \quad (8.15)$$

It is interesting to compare the present situation with the singular integrals arising from typical applications of the Hardy–Littlewood circle method. These are expressed as volumes that reflect the real density of solutions. It transpires that we will be able to provide a lower bound for $\sigma_\infty(G, H, V)$ which essentially matches the upper bound (8.15) without first approximating the sum by an integral. Nonetheless crucial use will be made of the fact that $\sigma_\infty(G, H, V)$ features a sum over points close to a non-singular real point on the variety (3.5).

For given k and u , let

$$N_M(k, u) := \# \{ \mathbf{p}, \mathbf{q}, \mathbf{s} \pmod{\Delta} : (8.11), (8.12) \text{ hold} \}, \quad (8.16)$$

where $\Delta = [M, u, k]$. We claim that

$$N_M(k, u) \ll_\xi \frac{\Delta^{3n-1+\xi}}{k}, \quad (8.17)$$

for any $\xi > 0$. To see this we write $A = \text{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{p}) \mathbf{N}_{K/L}(\mathbf{s}))$ for fixed \mathbf{p}, \mathbf{s} modulo Δ . Then the number of $\mathbf{q} \pmod{\Delta}$ contributing to $N_M(k, u)$ is at most

$$\sum_{\substack{N \ll \Delta^n \\ N \equiv A \pmod{u}}} \# \{ \mathbf{q} \in [1, \Delta]^n : 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{q}) = N \} \ll_\xi \frac{\Delta^{n+\xi/2}}{u}.$$

Moreover there are Δ^n possibilities for $\mathbf{s} \pmod{\Delta}$ and the number of available $\mathbf{p} \pmod{\Delta}$ is at most

$$\sum_{\substack{N \ll \Delta^n \\ N \equiv 0 \pmod{k^2}}} \# \{ \mathbf{p} \in [1, \Delta]^n : \mathbf{N}_{K/\mathbb{Q}}(\mathbf{p}) = N \} \ll_\xi \frac{\Delta^{n+\xi/2}}{k^2}. \quad (8.18)$$

It therefore follows that

$$N_M(k, u) \ll_\xi \frac{\Delta^{n+\xi/2}}{u} \cdot \Delta^n \cdot \frac{\Delta^{n+\xi/2}}{k^2} = \frac{\Delta^{3n+\xi}}{uk^2}.$$

Noting that $uk \gg \Delta$, we easily arrive at (8.17).

The extraction of $\sigma_\infty = \sigma_\infty(G, H, V)$ now follows on combining Lemmas 17 and 18 with (8.10) and (8.17). Putting these together, and assuming that $KQ \leq V$, we therefore deduce that

$$\begin{aligned} \mathcal{M} &= \sigma_\infty 2^\kappa M^n \sum_{k \leq K} \mu(k) \sum_{q \leq Q} \sum_{u|q} \frac{u\mu(q/u)}{\Delta^{3n}} N_M(k, u) \\ &\quad + O_\eta \left(H^n V^{2n+\eta} \left\{ \frac{Q^6}{H^{n/2}} + \frac{Q^{2+\eta}}{K} + \frac{Q^{2+\eta} K^\eta}{V} \right\} \right), \end{aligned}$$

where $N_M(k, u)$ is given by (8.16). Since $KQ \leq V$ it is clear that the error term is

$$\ll_{\eta} H^n V^{2n+2\eta} \left\{ \frac{Q^6}{H^{n/2}} + \frac{Q^2}{K} \right\}.$$

We now show that the summation over k can be extended to infinity with acceptable error. It follows from (8.15) and (8.17) that the error in so doing is

$$\ll_{\eta} G^{-2n} H^n V^{2n} \sum_{k>K} \sum_{q \leq Q} \sum_{u|q} \frac{u}{\Delta^{3n}} \cdot \frac{\Delta^{3n-1+\eta/2}}{k} \ll_{\eta} \frac{Q^{1+\eta} H^n V^{2n}}{K^{1-\eta}}.$$

This error term is subsumed by that above when $KQ \leq V$. Choosing

$$K = \frac{H^{n/2}}{Q},$$

the hypothesis (8.13) becomes $H^{n/2} \leq V$. Putting everything together we have therefore established the following result.

Lemma 19. — *Assume that $H^{n/2} \leq V$. Then we have*

$$\mathcal{M} = 2^{\kappa} M^n \sigma_{\infty} \mathfrak{S}(Q) + O_{\eta} \left(Q^6 H^{n/2} V^{2n+2\eta} \right),$$

where $\sigma_{\infty} = \sigma_{\infty}(G, H, V)$ is given by (8.14) and

$$\mathfrak{S}(Q) := \sum_{q \leq Q} \sum_{k=1}^{\infty} \mu(k) \sum_{u|q} \frac{u \mu(q/u)}{\Delta^{3n}} N_M(k, u),$$

with $\Delta = [M, u, k]$ and $N_M(k, u)$ given by (8.16).

9. The singular series and integral

It remains to produce satisfactory lower bounds for the quantities $\mathfrak{S}(Q)$ and σ_{∞} . For the former our strategy will be to show that as Q tends to infinity the sum $\mathfrak{S}(Q)$ converges to a limit \mathfrak{S} which is a product of local factors. We will then show that each of these factors is positive. In view of (8.17) the sum

$$f(q) := \sum_{k=1}^{\infty} \mu(k) \sum_{u|q} \frac{u \mu(q/u)}{\Delta^{3n}} N_M(k, u)$$

is absolutely convergent. We rearrange it as

$$f(q) = \sum_{u|q} u \mu(q/u) \sum_{k=1}^{\infty} \mu(k) \Delta^{-3n} N_M(k, u)$$

and write $k = k_1 k_2$ where $k_1 \mid uM$ and $\gcd(k_2, uM) = 1$. Then $[M, u, k] = [M, u] k_2$ and

$$f(q) = \sum_{u|q} u \mu(q/u) \sum_{k_1 \mid uM} \frac{\mu(k_1)}{[M, u]^{3n}} \sum_{k_2} \frac{\mu(k_2)}{k_2^{3n}} N_M(k_1 k_2, u).$$

The function $N_M(k_1 k_2, u)$ factors as

$$N_M(k_1 k_2, u) = N_M(k_1, u) \# \left\{ \mathbf{p}, \mathbf{q}, \mathbf{s} \pmod{k_2} : k_2 \mid \mathbf{N}_{K/L}(\mathbf{p}) \right\},$$

where $\Delta = [M, u]$ in $N_M(k_1, u)$. We write

$$R(k) := \# \{ \mathbf{p} \pmod{k} : k \mid \mathbf{N}_{K/L}(\mathbf{p}) \}$$

and observe that

$$R(k) \ll_{\xi} k^{n-2+\xi} \quad \text{and} \quad R(k) < k^n \text{ for } k > 1, \quad (9.1)$$

by the argument used for (8.18), and the fact that $\mathbf{N}_{K/\mathbb{Q}}(1, 0, 0, \dots, 0) = 1$. It follows that

$$\begin{aligned} \sum_{k_2} \frac{\mu(k_2)}{k_2^{3n}} N_M(k_1 k_2, u) &= N_M(k_1, u) \sum_{k_2} \frac{\mu(k_2)}{k_2^n} R(k_2) \\ &= c \prod_{p \mid uM} \left(1 - \frac{R(p)}{p^n} \right)^{-1} N_M(k_1, u), \end{aligned}$$

with

$$c = \prod_p \left(1 - \frac{R(p)}{p^n} \right) > 0. \quad (9.2)$$

We therefore see that $f(q) = c f_0(q, M)$ where

$$f_0(q, M) = \sum_{u \mid q} u \mu(q/u) \sum_{k \mid uM} \frac{\mu(k)}{[M, u]^{3n}} \prod_{p \mid uM} \left(1 - \frac{R(p)}{p^n} \right)^{-1} N_M(k, u) \quad (9.3)$$

is a multiplicative function in the two variables q and M . We write $q = \prod p^\alpha$ and $M = \prod p^\mu$, where $\mu = v_p(M)$ vanishes for all but finitely many primes. Then

$$\sum_{q=1}^{\infty} |f(q)| = c \prod_p \sum_{\alpha=0}^{\infty} |f_0(p^\alpha, p^\mu)|. \quad (9.4)$$

Providing that this product converges we will be able to deduce that

$$\lim_{Q \rightarrow \infty} \mathfrak{S}(Q) = \mathfrak{S}$$

exists. Moreover we will have

$$\mathfrak{S} = c \prod_p \sigma_p$$

with

$$\sigma_p = \sum_{\alpha=0}^{\infty} f_0(p^\alpha, p^\mu), \quad (9.5)$$

and in order to prove that $\mathfrak{S} > 0$ it will suffice to show that $\sigma_p > 0$ for every prime p .

Our treatment of the singular series will depend on the following two lemmas, which we will prove later in this section. It will be convenient to write \mathbf{x} for the vector $(\mathbf{p}, \mathbf{q}, \mathbf{s})$ and \mathbf{x}_0 for $(\mathbf{v}^{(M)}, \mathbf{w}^{(M)}, \mathbf{u}^{(M)})$. Moreover we shall write $F(\mathbf{x}) = 2^{-\kappa} F(\mathbf{p}; \mathbf{q}; \mathbf{s})$ and $\mathbf{N}(\mathbf{x}) = \mathbf{N}_{K/L}(\mathbf{p})$.

Lemma 20. — *Suppose that $p^\mu \parallel M$. For any $\beta \geq \max\{\mu, 1\}$ define*

$$M(p^\beta, p^\mu) := \# \{ \mathbf{x} \pmod{p^\beta} : \mathbf{x} \equiv \mathbf{x}_0 \pmod{p^\mu}, p^\beta \mid F(\mathbf{x}), p \nmid \mathbf{N}(\mathbf{x}) \}.$$

Then we will have

$$M(p, 1) = p^{3n-1} + O(p^{3n-5/2}). \quad (9.6)$$

Lemma 21. — *For any prime p and $\alpha \geq 2$ we have*

$$f_0(p^\alpha, p^\mu) \ll (2\alpha + 1)^{3n} p^{\alpha - 3[\alpha/2] - 2}.$$

We now prove that the product (9.4) converges. Lemma 21 yields

$$\sum_{\alpha=2}^{\infty} |f_0(p^\alpha, p^\mu)| \ll \sum_{\alpha=2}^{\infty} (2\alpha + 1)^{3n} p^{\alpha - 3[\alpha/2] - 2} \ll p^{-2}.$$

We note also that μ is non-zero only for the primes in S , and that $f_0(1, 1) = 1$. Thus a bound $f_0(p, 1) \ll p^{-3/2}$ will suffice to establish absolute convergence. In general we have

$$\begin{aligned} \sum_{\alpha \leq \beta} f_0(p^\alpha, p^\mu) &= \sum_{q|p^\beta} \sum_{u|q} u \mu(q/u) \sum_{k|up^\mu} \frac{\mu(k)}{[p^\mu, u]^{3n}} \prod_{p|up^\mu} \left(1 - \frac{R(p)}{p^n}\right)^{-1} N_{p^\mu}(k, u) \\ &= \sum_{u|p^\beta} u \sum_{k|up^\mu} \frac{\mu(k)}{[p^\mu, u]^{3n}} \prod_{p|up^\mu} \left(1 - \frac{R(p)}{p^n}\right)^{-1} N_{p^\mu}(k, u) \sum_{q|p^\beta: u|q} \mu(q/u), \end{aligned}$$

in view of (9.3). The final sum over q vanishes unless $u = p^\beta$, and if $\beta \geq \max\{\mu, 1\}$ our expression reduces to

$$\begin{aligned} \sum_{\alpha \leq \beta} f_0(p^\alpha, p^\mu) &= p^{-(3n-1)\beta} \left(1 - \frac{R(p)}{p^n}\right)^{-1} \sum_{k|p} \mu(k) N_{p^\mu}(k, p^\beta) \\ &= p^{-(3n-1)\beta} \left(1 - \frac{R(p)}{p^n}\right)^{-1} M(p^\beta, p^\mu). \end{aligned} \tag{9.7}$$

Taking $\beta = 1$ and $\mu = 0$ we obtain

$$f_0(p, 1) = p^{1-3n} \left(1 - \frac{R(p)}{p^n}\right)^{-1} \left\{ M(p, 1) - p^{3n-1} \left(1 - \frac{R(p)}{p^n}\right) \right\}.$$

The required estimate $f_0(p, 1) \ll p^{-3/2}$ now follows from Lemma 20, since $R(p)/p^n \ll p^{-3/2}$ by (9.1). This completes the proof of absolute convergence.

We turn now to proof that $\sigma_p > 0$ for every prime p . By (9.7) we have

$$\sigma_p = \lim_{\beta \rightarrow \infty} \sum_{\alpha \leq \beta} f_0(p^\alpha, p^\mu) = \left(1 - \frac{R(p)}{p^n}\right)^{-1} \lim_{\beta \rightarrow \infty} p^{-(3n-1)\beta} M(p^\beta, p^\mu).$$

Thus $\sigma_p > 0$ providing that $M(p^\beta, p^\mu) \gg_{p, \mu} p^{(3n-1)\beta}$ as $\beta \rightarrow \infty$.

Using Hensel's lemma it will follow that $\sigma_p > 0$ if the variety

$$F(\mathbf{v}; \mathbf{w}; \mathbf{u}) = \text{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{u}) \mathbf{N}_{K/L}(\mathbf{v})) - 2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) = 0,$$

introduced in (3.5), has a non-singular point over \mathbb{Z}_p which satisfies the constraints

$$p \nmid \mathbf{N}_{K/L}(\mathbf{v}) \quad \text{and} \quad (\mathbf{v}, \mathbf{w}, \mathbf{u}) \equiv (\mathbf{v}^{(M)}, \mathbf{w}^{(M)}, \mathbf{u}^{(M)}) \pmod{M}. \tag{9.8}$$

The function $M(p, 1)$ counts points over \mathbb{F}_p which lie on the above variety and satisfy the constraints (9.8), whether they are non-singular or not. The number of singular points will be $O(p^{3n-2})$, whence the estimate (9.6) shows that there will be a suitable non-singular point providing that $p \geq p_0$, say. We can arrange that the set S includes all primes $p < p_0$, so that

Lemma 3 ensures the existence of p -adic integer solutions with $\mathbf{v} \equiv (1, 0, \dots, 0) \pmod{p}$ and $\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) \neq 0$. It follows that $p \nmid \mathbf{N}_{K/L}(\mathbf{v})$. Finally we prove, by contradiction, that any such point must be non-singular. For otherwise we would have

$$\nabla_{\mathbf{w}} F(\mathbf{v}; \mathbf{w}; \mathbf{u}) = -2\nabla_{\mathbf{w}} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) = \mathbf{0}$$

where $\nabla_{\mathbf{w}} = (\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_n})$. It would then follow from Euler's identity that

$$\mathbf{w} \cdot \nabla_{\mathbf{w}} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) = n \mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}) = 0,$$

a contradiction. Thus we have suitable non-singular points for every prime p , and hence it follows that $\sigma_p > 0$ for all primes p .

We remark that, by combining (9.2), (9.5) and (9.7), we have

$$\mathfrak{S} = \prod_p \sigma_p^*,$$

with

$$\sigma_p^* := \lim_{\beta \rightarrow \infty} p^{-(3n-1)\beta} M(p^\beta, p^\mu).$$

Thus \mathfrak{S} is a standard product of local densities.

It remains to establish Lemmas 20 and 21. To handle Lemma 20 we observe that the variety defined by $F(\mathbf{x}) = 0$ takes the simple form

$$X_1 \cdots X_n + X_{n+1} \cdots X_{2n} + X_{2n+1} \cdots X_{3n} = 0$$

over $\overline{\mathbb{F}_p}$. This makes it clear that we have a hypersurface of projective dimension $3n-2$, whose singular locus has projective dimension $3n-7$. Here we use the fact that the singular locus consists of points where two or more coordinates vanish from each of the sets $\{X_1, \dots, X_n\}$, $\{X_{n+1}, \dots, X_{2n}\}$, and $\{X_{2n+1}, \dots, X_{3n}\}$. According to the result of Hooley [19] the number of projective points modulo p differs from $(p^{3n-1} - 1)/(p - 1)$ by an amount $O(p^{3n-4})$, whence the number of points in $\mathbb{A}^n(\mathbb{F}_p)$ is $p^{3n-1} + O(p^{3n-3})$. It follows that

$$M(p, 1) = p^{3n-1} + O(p^{3n-3}) - \#\{\mathbf{x} \pmod{p} : p \mid F(\mathbf{x}), p \mid \mathbf{N}(\mathbf{x})\}.$$

In the set on the right we have

$$p \mid \text{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{p}) \mathbf{N}_{K/L}(\mathbf{q})) - 2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})$$

and

$$p \mid \mathbf{N}_{K/L}(\mathbf{p}),$$

from which it follows that $p \mid 2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})$. It follows, by the argument leading to (8.18) that the number of possible $\mathbf{p} \pmod{p}$ is $O_\xi(p^{n-2+\xi})$ and that the number of possible $\mathbf{s} \pmod{p}$ is $O_\xi(p^{n-1+\xi})$. We then see that

$$\#\{\mathbf{x} \pmod{p} : p \mid F(\mathbf{x}), p \mid \mathbf{N}(\mathbf{x})\} \ll_\xi p^{3n-3+2\xi}$$

for any fixed $\xi > 0$, which gives us the required bound (9.6).

Turning to the proof of Lemma 21 we will begin by supposing that

$$\alpha \geq \max\{2\mu - 1, 2\}. \quad (9.9)$$

It follows that $\alpha > \max\{\mu, 1\}$, whence (9.7) yields

$$f_0(p^\alpha, p^\mu) = \frac{p^{-(3n-1)\alpha}}{1 - R(p)p^{-n}} (M(p^\alpha, p^\mu) - p^{3n-1} M(p^{\alpha-1}, p^\mu)). \quad (9.10)$$

It will be appropriate to observe at this point that $1 - R(p)/p^n \gg 1$, which follows from (9.1).

We proceed to compare $M(p^e, p^\mu)$ with $M(p^{e+1}, p^\mu)$, using Hensel lifting. For $t < e$ we define

$$S_t(p^e, p^\mu) := \{\mathbf{x} \pmod{p^e} : \mathbf{x} \equiv \mathbf{x}_0 \pmod{p^\mu}, p^e \mid F(\mathbf{x}), p \nmid \mathbf{N}(\mathbf{x}), p^t \mid \nabla F(\mathbf{x})\}.$$

When $t < e/2$ and $t \leq e - \max\{\mu, 1\}$ one sees that if $\mathbf{x} \in S_t(p^e, p^\mu)$ then $\mathbf{x} + p^{e-t}\mathbf{y} \in S_t(p^e, p^\mu)$ for all $\mathbf{y} \pmod{p^t}$. Thus $S_t(p^e, p^\mu)$ is composed of cosets modulo p^{e-t} . Moreover $\mathbf{x} + p^{e-t}\mathbf{y}$ will be in $S_t(p^{e+1}, p^\mu)$ for exactly p^{3n-1} choices of $\mathbf{y} \pmod{p}$. It follows that each coset modulo p^{e-t} in $S_t(p^e, p^\mu)$ lifts to exactly p^{3n-1} cosets modulo p^{e+1-t} in $S_t(p^{e+1}, p^\mu)$, and hence that

$$\#S_t(p^{e+1}, p^\mu) = p^{3n-1} \#S_t(p^e, p^\mu) \quad (9.11)$$

for $t < e/2$ and $t \leq e - \max\{\mu, 1\}$.

We now write

$$T(p^t) = \#\{\mathbf{x} \pmod{p^t} : p^t \mid \nabla F(\mathbf{x})\},$$

whence

$$\#\{\mathbf{x} \pmod{p^e} : p^t \mid \nabla F(\mathbf{x})\} = p^{3n(e-t)} T(p^t)$$

for $t \leq e$. Then for any non-negative integer $\tau \leq e$ we have

$$M(p^e, p^\mu) = \sum_{0 \leq t < \tau} \#S_t(p^e, p^\mu) + \#\left\{\mathbf{x} \pmod{p^e} : \begin{array}{l} \mathbf{x} \equiv \mathbf{x}_0 \pmod{p^\mu}, \\ p^e \mid F(\mathbf{x}), \\ p \nmid \mathbf{N}(\mathbf{x}), p^\tau \mid \nabla F(\mathbf{x}) \end{array}\right\},$$

whence

$$\left| M(p^e, p^\mu) - \sum_{0 \leq t < \tau} \#S_t(p^e, p^\mu) \right| \leq \#\{\mathbf{x} \pmod{p^e} : p^\tau \mid \nabla F(\mathbf{x})\} = p^{3n(e-\tau)} T(p^\tau).$$

Similarly we have

$$\left| M(p^{e+1}, p^\mu) - \sum_{0 \leq t < \tau} \#S_t(p^{e+1}, p^\mu) \right| \leq p^{3n(e+1-\tau)} T(p^\tau).$$

We therefore wish to use (9.11) for every value $t < \tau$. This will require that $\tau - 1 < e/2$ and $\tau - 1 \leq e - \max\{\mu, 1\}$. This condition is equivalent to requiring that $\tau - 1 \leq (e - 1)/2$ and $\tau - 1 \leq e - \max\{\mu, 1\}$, or that $\tau \leq (e + 1)/2$ and $\tau \leq e - \max\{\mu - 1, 0\}$. Thus if

$$\tau = \tau(e) = \min \left\{ e - \max\{\mu - 1, 0\}, \left\lceil \frac{e + 1}{2} \right\rceil \right\}$$

then (9.11) yields

$$|M(p^{e+1}, p^\mu) - p^{3n-1} M(p^e, p^\mu)| \leq 2p^{3n(e+1-\tau)} T(p^\tau).$$

We proceed to insert this bound into (9.10). If $e = \alpha - 1$ we find that $\tau(e) = \lfloor \alpha/2 \rfloor$ providing that $\alpha \geq \max\{2\mu - 1, 1\}$. In fact we have made the stronger assumption (9.9), and we deduce that

$$f_0(p^\alpha, p^\mu) \ll p^{\alpha - 3n \lfloor \alpha/2 \rfloor} T(p^{\lfloor \alpha/2 \rfloor}). \quad (9.12)$$

We have therefore reduced our problem to one of providing a suitable upper bound for $T(p^t)$.

We now recall that

$$F(\mathbf{x}) = 2^{-\kappa} (\text{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{p}) \mathbf{N}_{K/L}(\mathbf{q})) - 2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})).$$

If we set

$$G(\mathbf{p}, \mathbf{q}) = \text{Tr}_{L/\mathbb{Q}}(\delta \mathbf{N}_{K/L}(\mathbf{p}) \mathbf{N}_{K/L}(\mathbf{q}))$$

it then follows that

$$T(p^t) \leq T_1(p^t)T_2(p^t), \quad (9.13)$$

where

$$T_1(p^t) := \#\{(\mathbf{p}, \mathbf{q}) \pmod{p^t} : \nabla_{\mathbf{p}} G(\mathbf{p}, \mathbf{q}) \equiv \nabla_{\mathbf{q}} G(\mathbf{p}, \mathbf{q}) \equiv \mathbf{0} \pmod{p^t}\}$$

and

$$T_2(p^t) := \#\{\mathbf{s} \pmod{p^t} : \nabla \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s}) \equiv \mathbf{0} \pmod{p^t}\}.$$

We begin by explaining our estimation of $T_2(p^t)$. The treatment of $T_1(p^t)$ will then be in the same spirit, but a little more complicated. Over $\overline{\mathbb{Q}}$ we have

$$\mathbf{N}_{K/\mathbb{Q}}(\mathbf{s}) = \prod_{i=1}^n L_i(\mathbf{s})$$

for certain linearly independent linear forms

$$L_i(\mathbf{s}) = \sum_{j=1}^n c_{ij} s_j$$

with coefficients in the normal closure N , say, of K . Indeed our original choice of the basis $\omega_1, \dots, \omega_n$ ensures that the c_{ij} are algebraic integers. We now have

$$\frac{\partial}{\partial s_j} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s}) = \sum_{i=1}^n c_{ij} \mu_i,$$

where

$$\mu_i := L_1(\mathbf{s}) \cdots L_{i-1}(\mathbf{s}) L_{i+1}(\mathbf{s}) \cdots L_n(\mathbf{s}).$$

Let \mathbf{C} denote the matrix $(c_{ij})_{i,j \leq n}$ and write $D = \det \mathbf{C}$, so that D^2 is the discriminant of K . Suppose now that p^t divides $\nabla \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})$. Then if $\boldsymbol{\mu}$ is the column vector with elements μ_i we will have $\mathbf{C}\boldsymbol{\mu} \equiv \mathbf{0} \pmod{p^t}$. This divisibility relation may be interpreted in the ring of integers for N . We now pre-multiply by the matrix \mathbf{C}^{adj} , whose entries are algebraic integers, and use the fact that $\mathbf{C}^{\text{adj}}\mathbf{C} = (\det \mathbf{C})I$ to deduce that $D\boldsymbol{\mu} \equiv \mathbf{0} \pmod{p^t}$. We conclude that $p^t \mid D\mu_i$ for each index i , where divisibility is again within the ring of integers of N . Then $p^{tn} \mid D^n \prod_i \mu_i = D^n \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})^{n-1}$, and hence $p^{\lceil tn/(n-1) \rceil} \mid D^2 \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})$ whenever $p^t \mid \nabla \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})$.

Let

$$\sigma = \max \{ \lceil tn/(n-1) \rceil - v_p(D^2), 0 \}.$$

Since $t \leq \sigma \leq 2t$ we then have

$$\begin{aligned} T_2(p^t) &= p^{n(t-\sigma)} \#\{\mathbf{s} \pmod{p^\sigma} : p^t \mid \nabla \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})\} \\ &\leq p^{n(t-\sigma)} \#\{\mathbf{s} \in \mathbb{N}^n \cap (0, p^\sigma]^n : p^\sigma \mid \mathbf{N}_{K/\mathbb{Q}}(\mathbf{s})\}. \end{aligned}$$

Suppose that (p) splits over K as

$$(p) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \quad \text{with} \quad N_{K/\mathbb{Q}}(\mathfrak{p}_i) = p^{f_i},$$

for $1 \leq i \leq r$. Let $\alpha = \sum_{j=1}^n s_j \omega_j$, so that α is a non-zero element of \mathfrak{o}_K . If we now set $v_i = v_{\mathfrak{p}_i}(\alpha)$ we deduce that

$$p^\sigma \mid \prod_{i=1}^r N_{K/\mathbb{Q}}(\mathfrak{p}_i)^{v_i},$$

whence $\sigma \leq \sum f_i v_i$. It then follows that $\sigma \leq \sum f_i g_i$, where $g_i = \min\{v_i, \sigma\}$. Thus for each element \mathbf{s} there are non-negative integers $g_i \leq \sigma$ for which $\sum f_i g_i \geq \sigma$ and

$$\sum_{j=1}^n s_j \omega_j \in \mathfrak{p}_1^{g_1} \cdots \mathfrak{p}_r^{g_r}.$$

This condition restricts \mathbf{s} to a lattice Λ say, with $p^\sigma \mathbb{Z}^n \subseteq \Lambda \subseteq \mathbb{Z}^n$, and with $p^\sigma \mid \det(\Lambda)$. The number of choices for g_1, \dots, g_r is at most $(\sigma + 1)^r \leq (\sigma + 1)^n$. It therefore follows that

$$T_2(p^t) \leq p^{n(t-\sigma)} (\sigma + 1)^n p^{n\sigma-\sigma} \leq (2t + 1)^n p^{nt-\sigma} \ll (2t + 1)^n p^{nt - \lceil tn/(n-1) \rceil}, \quad (9.14)$$

since D is fixed.

We now examine $T_1(p^t)$ in a similar way. The form $G(\mathbf{p}, \mathbf{q})$ takes the shape

$$\delta \prod_{i=1}^{n/2} L_i(\mathbf{p}) \prod_{i=1}^{n/2} L_i(\mathbf{q}) + \delta^\sigma \prod_{i=n/2+1}^n L_i(\mathbf{p}) \prod_{i=n/2+1}^{n/2} L_i(\mathbf{q}).$$

If we replace $\delta L_1(\mathbf{p})$ by $L_1(\mathbf{p})$ and $\delta^\sigma L_{n/2+i}(\mathbf{p})$ by $L_{n/2+1}(\mathbf{p})$ we find that

$$\frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial p_j} = \sum_{i=1}^n c_{ij} \mu_i,$$

where

$$c_{ij} = \frac{\partial L_i(\mathbf{x})}{\partial x_j}$$

and

$$\mu_i = L_1(\mathbf{p}) \cdots L_{i-1}(\mathbf{p}) L_{i+1}(\mathbf{p}) \cdots L_{n/2}(\mathbf{p}) \prod_{i=1}^{n/2} L_i(\mathbf{q})$$

for $1 \leq i \leq n/2$ and

$$\mu_i = L_{n/2+1}(\mathbf{p}) \cdots L_{n/2+i-1}(\mathbf{p}) L_{n/2+i+1}(\mathbf{p}) \cdots L_n(\mathbf{p}) \prod_{i=n/2+1}^n L_i(\mathbf{q})$$

for $n/2 < i \leq n$. Thus $p^t \mid D\mu_i$ for $1 \leq i \leq n$, where D is the determinant of the matrix $\mathbf{C} = (c_{ij})_{i,j \leq n}$. In particular D is a non-zero algebraic integer. Taking the product for $i \leq n$ yields $p^{tn} \mid D^n \mathbf{N}_{K/\mathbb{Q}}(\mathbf{p})^{n/2-1} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{q})^{n/2}$. By symmetry we also obtain $p^{tn} \mid D^n \mathbf{N}_{K/\mathbb{Q}}(\mathbf{q})^{n/2-1} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{p})^{n/2}$, and hence

$$p^{2tn} \mid D^{2n} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{p})^{n-1} \mathbf{N}_{K/\mathbb{Q}}(\mathbf{q})^{n-1}.$$

It follows that $p^\sigma \mid \mathbf{N}_{K/\mathbb{Q}}(\mathbf{p}) \mathbf{N}_{K/\mathbb{Q}}(\mathbf{q})$ with

$$\sigma = \max \{ \lceil 2tn/(n-1) \rceil - v_p(D^4), 0 \}.$$

We can now complete the argument as before. This time there will be two sets of non-negative exponents g_i and g'_i , say, corresponding to \mathbf{p} and \mathbf{q} respectively, and such that $\sum_i f_i (g_i + g'_i) \geq \sigma$. Thus the factor $(\sigma + 1)^n$ must be replaced by $(\sigma + 1)^{2n}$. For each such set of exponents we find that (\mathbf{p}, \mathbf{q}) is restricted to a sublattice of \mathbb{Z}^{2n} of index at least p^σ , and we deduce as before that

$$T_1(p^t) \ll (4t + 1)^{2n} p^{2nt - \lceil 2tn/(n-1) \rceil}.$$

In view of (9.13) and (9.14) we deduce that

$$T(p^t) \ll (4t+1)^{3n} p^{3nt - \lceil tn/(n-1) \rceil - \lceil 2tn/(n-1) \rceil} \ll (4t+1)^{3n} p^{3nt-3t-2},$$

since if $m \in \mathbb{Z}$ we have $\lceil \theta \rceil \geq m+1$ for any real number $\theta > m$. It now follows from (9.12) that

$$f_0(p^\alpha, p^\mu) \ll (2\alpha+1)^{3n} p^{\alpha-3[\alpha/2]-2},$$

and this suffices for Lemma 21 when (9.9) holds. Finally, it is clear in fact that we still have this estimate if $2 \leq \alpha \leq 2\mu-1$, since if $\mu > 0$ then p belongs to the finite set of divisors of M . This therefore completes the proof of Lemma 21.

Our final task in this section is to establish a lower bound for σ_∞ to complement the upper bound in (8.15). For any $\mathbf{c} \in \mathbb{R}^2$ let $B(\mathbf{c}; \rho) \subset \mathbb{R}^2$ denote the box centered on \mathbf{c} with side length 2ρ . The final part of Lemma 9 implies that there exists $\rho \gg G^{-1}U^{n/2}$ such that

$$\omega(x_1, x_2) \gg G^{2-n}$$

for every $(x_1, x_2) \in B(\delta \mathbf{N}_{K/L}(U\mathbf{u}^{(\mathbb{R})}); \rho)$. Here we view $\delta \mathbf{N}_{K/L}(U\mathbf{u}^{(\mathbb{R})})$ as a vector (c_1, c_2) in \mathbb{R}^2 . In this way we deduce from (8.6) and (8.14) that

$$\sigma_\infty \gg G^{2-n} \sum_{\mathbf{w} \in \mathcal{W} \cap \mathbb{Z}^n} \sum_{\mathbf{v} \in \mathcal{V} \cap \mathbb{Z}^n} J(\mathbf{v}; \mathbf{w}), \quad (9.15)$$

where if $a_i = a_i(\mathbf{v})$ and $b = b(\mathbf{w})$ are given by (8.2) then

$$J(\mathbf{v}; \mathbf{w}) := \text{meas} \left\{ x \in \mathbb{R} : \left(a_2(\mathbf{v})x, -a_1(\mathbf{v})x + \frac{b(\mathbf{w})}{a_2(\mathbf{v})} \right) \in B(\mathbf{c}; \rho) \right\}.$$

The minimum distance from the line $(a_2x, -a_1x + b/a_2)$ to the point (c_1, c_2) is equal to $|a_1c_1 + a_2c_2 - b|/\sqrt{a_1^2 + a_2^2}$. Suppose now that we have points \mathbf{v} and \mathbf{w} satisfying

$$|\mathbf{v} - V\mathbf{v}^{(\mathbb{R})}| < \lambda G^{-1}V \quad \text{and} \quad |\mathbf{w} - W\mathbf{w}^{(\mathbb{R})}| < \lambda G^{-1}W \quad (9.16)$$

for some $\lambda \leq 1$. If we set

$$a_1^{(\mathbb{R})} = \text{Tr}_{L/\mathbb{Q}}(\mathbf{N}_{K/L}(\mathbf{v}^{(\mathbb{R})})), \quad a_2^{(\mathbb{R})} = \text{Tr}_{L/\mathbb{Q}}(\tau \mathbf{N}_{K/L}(\mathbf{v}^{(\mathbb{R})}))$$

and

$$b^{(\mathbb{R})} = 2\mathbf{N}_{K/\mathbb{Q}}(\mathbf{w}^{(\mathbb{R})}),$$

then

$$a_1(\mathbf{v}) = V^{n/2}a_1^{(\mathbb{R})} + O(\lambda G^{-1}V^{n/2}), \quad a_2(\mathbf{v}) = V^{n/2}a_2^{(\mathbb{R})} + O(\lambda G^{-1}V^{n/2})$$

and

$$b(\mathbf{w}) = W^n b^{(\mathbb{R})} + O(\lambda G^{-1}W^n).$$

Moreover

$$c_1 V^{n/2} a_1^{(\mathbb{R})} + c_2 V^{n/2} a_2^{(\mathbb{R})} - W^n b^{(\mathbb{R})} = F(V\mathbf{v}^{(\mathbb{R})}; W\mathbf{w}^{(\mathbb{R})}; U\mathbf{u}^{(\mathbb{R})}) = 0.$$

Thus $|a_1c_1 + a_2c_2 - b| = O(\lambda G^{-1}W^n)$, while $a_1^2 + a_2^2 \gg V^n$ since $a_1^{(\mathbb{R})}, a_2^{(\mathbb{R})}$ are not both zero. Since $\rho \gg G^{-1}U^{n/2}$, it follows that

$$\frac{|a_1c_1 + a_2c_2 - b|}{\sqrt{a_1^2 + a_2^2}} \leq \frac{\rho}{2}$$

providing that we take λ as a sufficiently small positive constant.

We now see that, for points (\mathbf{v}, \mathbf{w}) satisfying (9.16), the line $(a_2x, -a_1x + b/a_2)$ meets the disc $B(\delta \mathbf{N}_{K/L}(U\mathbf{u}^{(\mathbb{R})}); \rho)$ in a segment of length $\gg \rho$, so that

$$J(\mathbf{v}; \mathbf{w}) \gg \frac{\rho}{\max\{|a_1|, |a_2|\}} \gg \frac{\rho}{V^{n/2}} \gg G^{-1} H^{n/2}.$$

The number of available points (\mathbf{v}, \mathbf{w}) is $\gg G^{-2n} (VW)^n$ and we therefore conclude from (9.15) that

$$\sigma_\infty \gg G^{1-3n} H^n V^{2n}.$$

This should be compared with the upper bound (8.15).

Bringing together our lower bounds for \mathfrak{S} and σ_∞ in Lemma 19, we deduce that

$$\mathcal{M} \gg G^{1-3n} H^n V^{2n} + O_\eta \left(Q^6 H^{n/2} V^{2n+2\eta} \right),$$

provided that $H^{n/2} \leq V$. This therefore leads to the following conclusion.

Lemma 22. — *Let $G = \log V$. Assume that $H^{n/2} \leq V$ and*

$$Q \leq H^{n/12} V^{-\eta/2}.$$

Then we have

$$\mathcal{M}(G, H, V) \gg (\log V)^{1-3n} H^n V^{2n}.$$

Recalling (3.18) and our choice $G = \log V$, it is now time to select parameters Q, H, V such that $\mathcal{E}(G, H, V) = o(\mathcal{M}(G, H, V))$. We will choose $Q = H^{(n-1)/12}$, with which choice Lemma 22 implies that $\mathcal{M}(G, H, V) \gg (\log V)^{1-3n} H^n V^{2n}$, if

$$V^{6\eta} \leq H \leq V^{2/n}.$$

In line with Lemma 3 we let V run through large integers congruent to 1 modulo M . Next we choose $H_0 = 1 + M[V^{1/(10n^2)}]$, which is a positive integer congruent to 1 modulo M . But then $H = H_0^2$ has order $V^{1/(5n^2)}$ and so $Q = H^{(n-1)/12}$ satisfies the conditions of Lemma 16. This implies that the required estimate for $\mathcal{E}(G, H, V)$ holds and so completes the proof of Theorem 2.

References

- [1] H.-J. Bartels, Zur Arithmetik von Konjugationsklassen in algebraischen Gruppen. *J. Algebra* **70** (1981), 179–199.
- [2] H.-J. Bartels, Zur Arithmetik von Diedergruppenerweiterungen. *Math. Ann.* **256** (1981), 465–473.
- [3] J. Brüdern, Binary additive problems and the circle method, multiplicative sequences and convergent sieves. *Analytic number theory*, 91–132, Cambridge Univ. Press, Cambridge, 2009.
- [4] J.-L. Colliot-Thélène and P. Salberger, Arithmetic on some singular cubic hypersurfaces. *Proc. London Math. Soc.* **58** (1989), 519–549.
- [5] J.-L. Colliot-Thélène and J.J. Sansuc, La R -équivalence sur les tores. *Ann. Sci. École Norm. Sup.* **10** (1977), 175–229.
- [6] J.-L. Colliot-Thélène and J.J. Sansuc, Principal homogeneous spaces under flasque tori: applications. *J. Algebra* **106** (1987), 148–205.
- [7] J.-L. Colliot-Thélène and A.N. Skorobogatov, Descent on fibrations over \mathbb{P}_k^1 revisited. *Math. Proc. Camb. Phil. Soc.* **128** (2000), 383–393.

- [8] J.-L. Colliot-Thélène, D. Harari and A.N. Skorobogatov, Valeurs d'un polynôme à une variable représentés par une norme. *Number theory and algebraic geometry*, 69–89, London Math. Soc. Lecture Note Ser. **303** Cambridge Univ. Press, Cambridge, 2003.
- [9] J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer, Intersections of two quadrics and Châtelet surfaces, I. *J. reine angew. Math.* **373** (1987), 37–107; II. *J. reine angew. Math.* **374** (1987), 72–168.
- [10] J.-L. Colliot-Thélène and P. Swinnerton-Dyer, Hasse principle and weak approximation for pencils of Severi–Brauer and similar varieties. *J. reine angew. Math.* **453** (1994), 49–112.
- [11] J.-L. Colliot-Thélène, A.N. Skorobogatov and P. Swinnerton-Dyer, Rational points and zero-cycles on fibred varieties: Schinzel’s hypothesis and Salberger’s device. *J. reine angew. Math.* **495** (1998), 1–28.
- [12] E. Fouvry and H. Iwaniec, Gaussian primes. *Acta Arith.* **79** (1997), 249–287.
- [13] G. Gras, *Class field theory: from theory to practice*. Springer-Verlag, 2002.
- [14] B. Green and T. Tao, Linear equations in primes. *Annals of Math.* **171** (2010), 1753–1850.
- [15] B. Green, T. Tao and T. Ziegler, An inverse theorem for the Gowers $U^{s+1}[N]$ -norm. *Submitted*, 2010.
- [16] S. Gurak, On the Hasse norm principle. *J. reine angew. Math.* **299/300** (1978), 16–27.
- [17] D.R. Heath-Brown, The ternary Goldbach problem. *Rev. Mat. Iberoamericana* **1** (1985), 45–59.
- [18] D.R. Heath-Brown and A.N. Skorobogatov, Rational solutions of certain equations involving norms. *Acta Math.* **189** (2002), 161–177.
- [19] C. Hooley, On the number of points on a complete intersection over a finite field. *J. Number Theory* **38** (1991), 338–358.
- [20] M.N. Huxley, The large sieve inequality for algebraic number fields. *Mathematika* **15** (1968), 178–187.
- [21] H. Iwaniec and E. Kowalski, *Analytic number theory*. American Math. Soc. Colloq. Pub. **53**, American Math. Soc., 2004.
- [22] J.V. Linnik, *The dispersion method in binary additive problems*. American Math. Soc. Providence, R.I., 1963.
- [23] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, *J. reine angew. Math.* **327** (1981), 12–80.
- [24] V.E. Voskresenskiĭ, *Algebraic groups and their birational invariants* (translated from Russian by B. Kunyavskiĭ). Translations of Math. Monographs **179**, American Math. Soc. 1998.

September 2, 2011

T.D. BROWNING, School of Mathematics, University of Bristol, Bristol, BS8 1TW, United Kingdom

E-mail : t.d.browning@bristol.ac.uk

D.R. HEATH-BROWN, Mathematical Institute, University of Oxford, Oxford, OX1 3LB, United Kingdom

E-mail : rhb@maths.ox.ac.uk